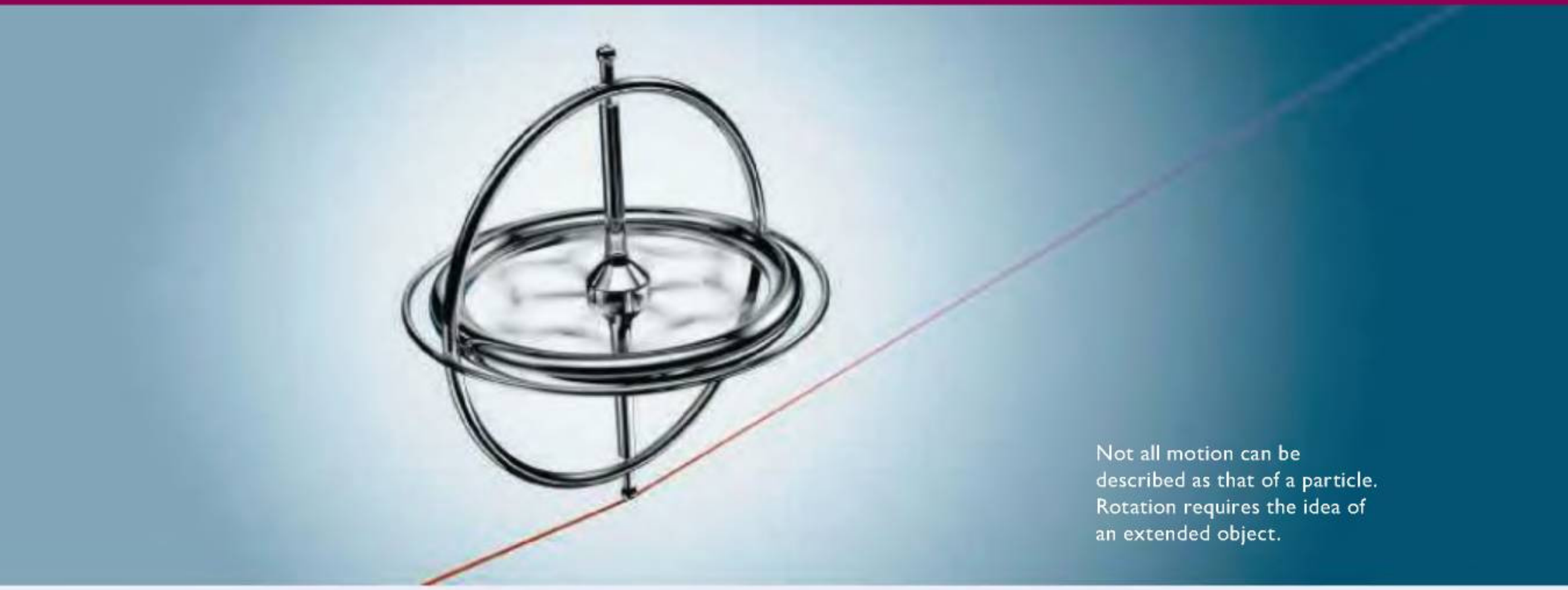
12 Rotation of a Rigid Body



Not all motion can be described as that of a particle. Rotation requires the idea of an extended object.

IN THIS CHAPTER, you will learn to understand and apply the physics of rotation.

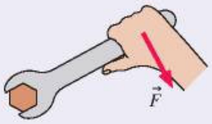
What is a rigid body?

An object whose size and shape don't change as it moves is called a rigid body. A rigid body is characterized by its moment of inertia I, which is the rotational equivalent of mass. We'll consider

- Rotation about an axle.
- Rolling without slipping.

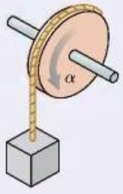
« LOOKING BACK Section 6.1 Equilibrium

What is torque?



Torque is the tendency or ability of a force to rotate an object about a pivot point. You'll learn that torque depends on both the force *and* where the force is applied. A longer wrench provides a larger torque.

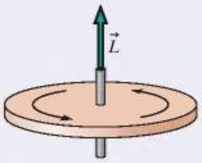
What does torque do?



Torque is to rotation what force is to linear motion. Torque τ causes an object to have angular acceleration. Newton's second law for rotation is $\alpha = \tau/I$. Much of rotational dynamics will look familiar because it is analogous to linear dynamics.

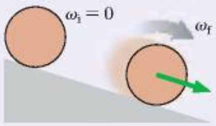
« LOOKING BACK Section 6.2 Newton's second law

What is angular momentum?



Angular momentum is to rotation what momentum is to linear motion. Angular momentum is an object's tendency to "keep rotating." Angular momentum $\vec{L}$ is a *vector* pointing along the rotation axis. You'll use angular momentum to understand the precession of a spinning top or gyroscope.

What is conserved in rotational motion?



The mechanical energy of a rotating object includes its rotational kinetic energy $\frac{1}{2}I\omega^2$. This is analogous to linear kinetic energy.

- Mechanical energy is conserved for frictionless, rotating systems.
- Angular momentum is conserved for isolated systems.

« LOOKING BACK Section 10.4 Conservation of energy; Section 11.2 Conservation of momentum

Why is rigid-body motion important?

The world is full of rotating objects, from windmill turbines to the gyroscopes used in navigation. The wheels on your bicycle or car roll without slipping. Scientists investigate rotating molecules and rotating galaxies. No understanding of motion is complete without understanding rotational motion, and this chapter will develop the tools you need. We'll also expand our understanding of equilibrium by exploring the conditions under which objects *don't* rotate.

12.1 Rotational Motion

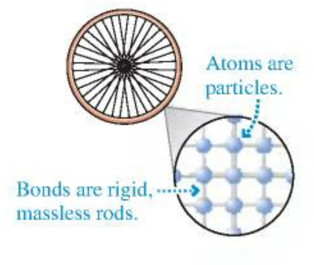
Thus far, our study of motion has focused almost exclusively on the *particle model,* in which an object is represented as a mass at a single point in space. As we expand our study of motion to rotation, we need to consider *extended objects* whose size and shape *do* matter. Thus this chapter will be based on the **rigid-body model:**

MODEL 12.1

Rigid-body model

A **rigid body** is an extended object whose size and shape do not change as it moves.

- Particle-like atoms are held together by rigid massless rods.
- A rigid body cannot be stretched, compressed, or deformed. All points on the body have the same angular velocity and angular acceleration.
- Limitations: Model fails if an object changes shape or is deformed.



Exercise 1

FIGURE 12.1 illustrates the three basic types of motion of a rigid body: **translational motion, rotational motion,** and **combination motion.**

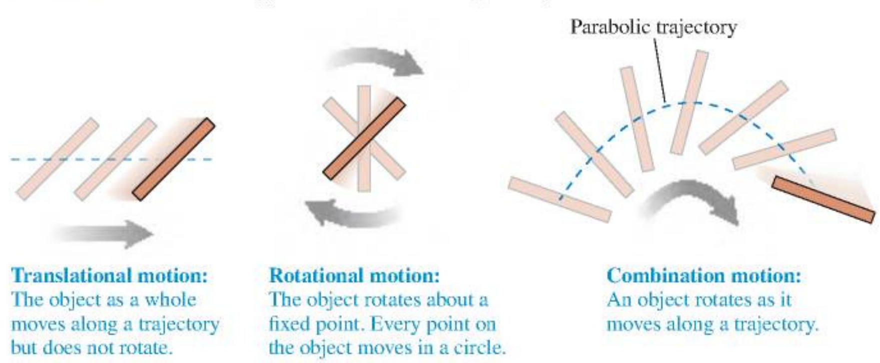


FIGURE 12.1 Three basic types of motion of a rigid body.

Brief Review of Rotational Kinematics

Rotation is an extension of circular motion, so we begin with a brief summary of Chapter 4. **A review of « Sections 4.4–4.6 is highly recommended. FIGURE 12.2** shows a wheel rotating on an axle. Its angular velocity

$$\omega = \frac{d\theta}{dt} \tag{12.1}$$

is the rate at which the wheel rotates. The SI units of ω are radians per second (rad/s), but revolutions per second (rev/s) and revolutions per minute (rpm) are frequently used. Notice that all points have equal angular velocities, so we can refer to the angular velocity ω *of the wheel.*

If the wheel is speeding up or slowing down, its angular acceleration is

$$\alpha = \frac{d\omega}{dt} \tag{12.2}$$

The units of angular acceleration are rad/s^2. Angular acceleration is the *rate* at which the angular velocity ω changes, just as the linear acceleration is the rate at which the

FIGURE 12.2 Rotational motion.

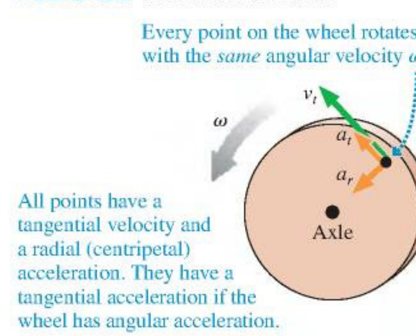


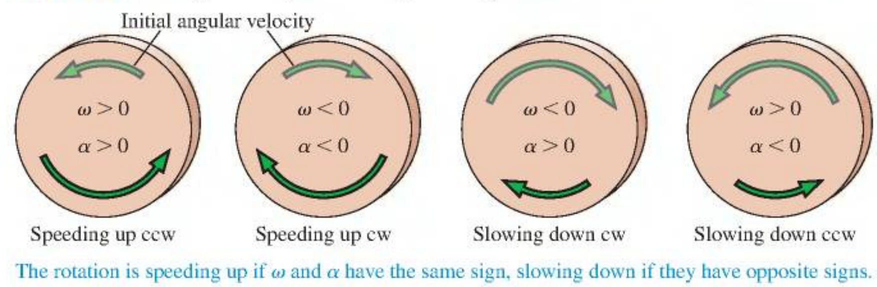
TABLE 12.1 Rotational kinematics for constant angular acceleration

$\omega_f = \omega_i + \alpha\,\Delta t$
$\theta_f = \theta_i + \omega_i\,\Delta t + \frac{1}{2}\alpha(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha\,\Delta\theta$

linear velocity v changes. **TABLE 12.1** summarizes the kinematic equations for rotation with constant angular acceleration.

FIGURE 12.3 reminds you of the sign conventions for angular velocity and acceleration. They will be especially important in the present chapter. Be careful with the sign of α. Just as with linear acceleration, positive and negative values of α can't be interpreted as simply "speeding up" and "slowing down."

FIGURE 12.3 The signs of angular velocity and angular acceleration.



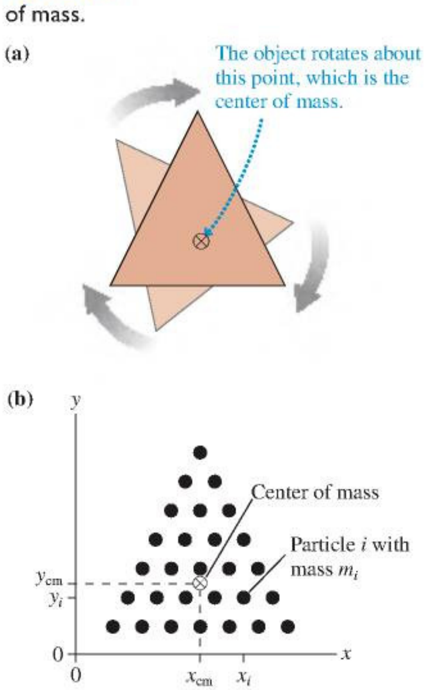
A point at distance r from the rotation axis has instantaneous velocity and acceleration, shown in Figure 12.2, given by

$$\begin{aligned} v_r &= 0 \qquad & a_r &= \frac{v_t^2}{r} = \omega^2 r \\ v_t &= r\omega \qquad & a_t &= r\alpha \end{aligned} \tag{12.3}$$

The sign convention for ω implies that v_t and a_t are positive if they point in the counterclockwise (ccw) direction, negative if they point in the clockwise (cw) direction.

12.2 Rotation About the Center of Mass

FIGURE 12.4 Rotation about the center of mass.



Imagine yourself floating in a space capsule deep in space. Suppose you take an object like that shown in **FIGURE 12.4a** and spin it so that it simply rotates but has no translational motion as it floats beside you. *About what point does it rotate?* That is the question we need to answer.

An unconstrained object (i.e., one not on an axle or a pivot) on which there is no net force rotates about a point called the **center of mass.** The center of mass remains motionless while every other point in the object undergoes circular motion around it. You need not go deep into space to demonstrate rotation about the center of mass. If you have an air table, a flat object rotating on the air table rotates about its center of mass.

To locate the center of mass, **FIGURE 12.4b** models the object as a set of particles numbered $i = 1, 2, 3, \ldots$. Particle i has mass m_i and is located at position (x_i, y_i). We'll prove later in this section that the center of mass is located at position

$$\begin{aligned} x_{cm} &= \frac{1}{M}\sum_i m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \\ y_{cm} &= \frac{1}{M}\sum_i m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \end{aligned} \tag{12.4}$$

where $M = m_1 + m_2 + m_3 + \cdots$ is the object's total mass.

Let's see if Equations 12.4 make sense. Suppose you have an object consisting of N particles, all with the same mass m. That is, $m_1 = m_2 = \cdots = m_N = m$. We can factor the m out of the numerator, and the denominator becomes simply Nm. The m cancels, and the x-coordinate of the center of mass is

$$x_{cm} = \frac{x_1 + x_2 + \cdots + x_N}{N} = x_{average}$$

In this case, x_{cm} is simply the *average* x-coordinate of all the particles. Likewise, y_{cm} will be the average of all the y-coordinates.

This *does* make sense! If the particle masses are all the same, the center of mass should be at the center of the object. And the "center of the object" is the average position of all the particles. To allow for *unequal* masses, Equations 12.4 are called *weighted averages*. Particles of higher mass count more than particles of lower mass, but the basic idea remains the same. **The center of mass is the mass-weighted center of the object.**

EXAMPLE 12.1 The center of mass of a barbell

A barbell consists of a 500 g ball and a 2.0 kg ball connected by a massless 50-cm-long rod.

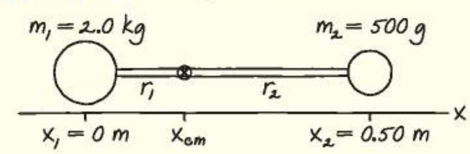
a. Where is the center of mass?

b. What is the speed of each ball if they rotate about the center of mass at 40 rpm?

MODEL Model the barbell as a rigid body.

VISUALIZE FIGURE 12.5 shows the two masses. We've chosen a coordinate system in which the masses are on the x-axis with the 2.0 kg mass at the origin.

FIGURE 12.5 Finding the center of mass.



SOLVE a. We can use Equations 12.4 to calculate that the center of mass is

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$
$$= \frac{(2.0\text{ kg})(0.0\text{ m}) + (0.50\text{ kg})(0.50\text{ m})}{2.0\text{ kg} + 0.50\text{ kg}} = 0.10\text{ m}$$

$y_{cm} = 0$ because all the masses are on the x-axis. The center of mass is 20% of the way from the 2.0 kg ball to the 0.50 kg ball.

b. Each ball rotates about the center of mass. The radii of the circles are $r_1 = 0.10$ m and $r_2 = 0.40$ m. The tangential velocities are $(v_i)_t = r_i\omega$, but this equation requires ω to be in rad/s. The conversion is

$$\omega = 40\,\frac{\text{rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ s}} \times \frac{2\pi\text{ rad}}{1\text{ rev}} = 4.19\text{ rad/s}$$

Consequently,

$$(v_1)_t = r_1\omega = (0.10\text{ m})(4.19\text{ rad/s}) = 0.42\text{ m/s}$$
$$(v_2)_t = r_2\omega = (0.40\text{ m})(4.19\text{ rad/s}) = 1.68\text{ m/s}$$

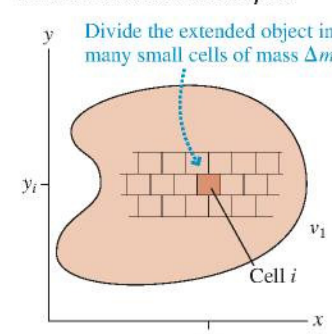
ASSESS The center of mass is closer to the heavier ball than to the lighter ball. We expected this because x_{cm} is a mass-weighted average of the positions. But the lighter mass moves faster because it is farther from the rotation axis.

Finding the Center of Mass by Integration

For any realistic object, carrying out the summations of Equations 12.4 over all the atoms in the object is not practical. Instead, as FIGURE 12.6 shows, we can divide an extended object into many small cells or boxes, each with the same very small mass Δm. We will number the cells 1, 2, 3, … , just as we did the particles. Cell i has coordinates (x_i, y_i) and mass $m_i = \Delta m$. The center-of-mass coordinates are then

$$x_{cm} = \frac{1}{M}\sum_i x_i\,\Delta m \quad\text{and}\quad y_{cm} = \frac{1}{M}\sum_i y_i\,\Delta m$$

FIGURE 12.6 Calculating the center of mass of an extended object.



Now, as you might expect, we'll let the cells become smaller and smaller, with the total number increasing. As each cell becomes infinitesimally small, we can replace Δm with dm and the sum by an integral. Then

$$x_{cm} = \frac{1}{M}\int x\,dm \quad\text{and}\quad y_{cm} = \frac{1}{M}\int y\,dm \qquad (12.5)$$

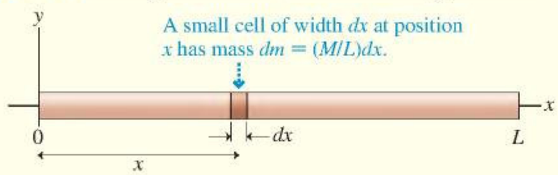
Equations 12.5 are a formal definition of the center of mass, but they are *not* ready to integrate in this form. First, integrals are carried out over *coordinates*, not over masses. Before we can integrate, we must replace dm by an equivalent expression involving a coordinate differential such as dx or dy. Second, no limits of integration have been specified. The procedure for using Equations 12.5 is best shown with an example.

EXAMPLE 12.2 | The center of mass of a rod

Find the center of mass of a thin, uniform rod of length L and mass M. Use this result to find the tangential acceleration of one tip of a 1.60-m-long rod that rotates about its center of mass with an angular acceleration of 6.0 rad/s^2.

VISUALIZE FIGURE 12.7 shows the rod. We've chosen a coordinate system such that the rod lies along the x-axis from 0 to L. Because the rod is "thin," we'll assume that $y_{cm} = 0$.

FIGURE 12.7 Finding the center of mass of a long, thin rod.



SOLVE Our first task is to find x_{cm}, which lies somewhere on the x-axis. To do this, we divide the rod into many small cells of mass dm. One such cell, at position x, is shown. The cell's width is dx. Because the rod is *uniform,* the mass of this little cell is the *same fraction* of the total mass M that dx is of the total length L. That is,

$$\frac{dm}{M} = \frac{dx}{L}$$

Consequently, we can express dm in terms of the coordinate differential dx as

$$dm = \frac{M}{L}dx$$

NOTE The change of variables from dm to the differential of a coordinate is *the* key step in calculating the center of mass.

With this expression for dm, Equation 12.5 for x_{cm} becomes

$$x_{cm} = \frac{1}{M}\left(\frac{M}{L}\int x\,dx\right) = \frac{1}{L}\int_0^L x\,dx$$

where in the last step we've noted that summing "all the mass in the rod" means integrating from $x = 0$ to $x = L$. This is a straightforward integral to carry out, giving

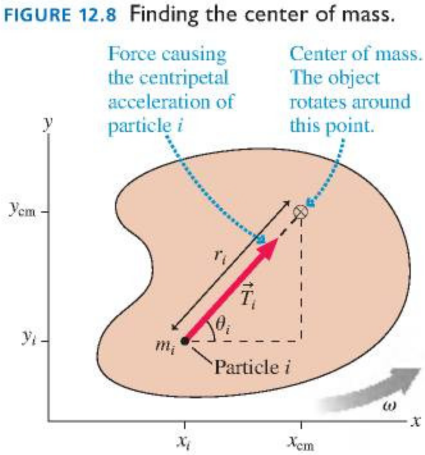
$$x_{cm} = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{1}{L}\left[\frac{L^2}{2} - 0\right] = \tfrac{1}{2}L$$

The center of mass is at the center of the rod, as you probably guessed. For a 1.60-m-long rod, each tip of the rod rotates in a circle with $r = \frac{1}{2}L = 0.80$ m. The tangential acceleration, the rate at which the tip is speeding up, is

$$a_t = r\alpha = (0.80\text{ m})(6.0\text{ rad/s}^2) = 4.8\text{ m/s}^2$$

NOTE For any symmetrical object of uniform density, the center of mass is at the physical center of the object.

FIGURE 12.8 Finding the center of mass.



To see where the center-of-mass equations come from, FIGURE 12.8 shows an object rotating about its center of mass. Particle i is moving in a circle, so it *must* have a centripetal acceleration. Acceleration requires a force, and this force is due to tension in the molecular bonds that hold the object together. Force $\vec{T}_i$ on particle i has magnitude

$$T_i = m_i(a_i)_r = m_i r_i \omega^2 \tag{12.6}$$

where we used Equation 12.3 for a_r. All points in a rigid rotating object have the *same* angular velocity, so ω doesn't need a subscript.

At every instant of time, the internal tension forces are all paired as action/reaction forces, equal in magnitude but opposite in direction, so the sum of all the tension forces must be zero. That is, $\sum \vec{T}_i = \vec{0}$. The x-component of this sum is

$$\sum_i (T_i)_x = \sum_i T_i \cos\theta_i = \sum_i (m_i r_i \omega^2)\cos\theta_i = 0 \tag{12.7}$$

You can see from Figure 12.8 that $\cos\theta_i = (x_{cm} - x_i)/r_i$. Thus

$$\sum_i (T_i)_x = \sum_i (m_i r_i \omega^2)\frac{x_{cm} - x_i}{r_i} = \left(\sum_i m_i x_{cm} - \sum_i m_i x_i\right)\omega^2 = 0 \tag{12.8}$$

This equation will be true if the term in parentheses is zero. x_{cm} is a constant, so we can bring it outside the summation to write

$$\sum_i m_i x_{cm} - \sum_i m_i x_i = \left(\sum_i m_i\right)x_{cm} - \sum_i m_i x_i = Mx_{cm} - \sum_i m_i x_i = 0 \tag{12.9}$$

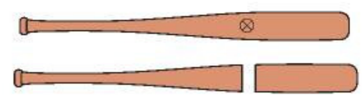
where we used the fact that $\sum m_i$ is simply the object's total mass M. Solving for x_{cm}, we find the x-coordinate of the object's center of mass to be

$$x_{cm} = \frac{1}{M}\sum_i m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{12.10}$$

This was Equation 12.4. The y-equation is found similarly.

STOP TO THINK 12.1 A baseball bat is cut into two pieces at its center of mass. Which end is heavier?

a. The handle end (left end).
b. The hitting end (right end).
c. The two ends weigh the same.



12.3 Rotational Energy

A rotating rigid body—whether it's rotating freely about its center of mass or constrained to rotate on an axle—has kinetic energy because all atoms in the object are in motion. The kinetic energy due to rotation is called **rotational kinetic energy.**

FIGURE 12.9 shows a few of the particles making up a solid object that rotates with angular velocity ω. Particle i, which rotates in a circle of radius r_i, moves with speed $v_i = r_i\omega$. The object's rotational kinetic energy is the sum of the kinetic energies of each of the particles:

$$K_{rot} = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \cdots$$
$$= \tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2 + \cdots = \tfrac{1}{2}\left(\sum_i m_ir_i^2\right)\omega^2 \qquad (12.11)$$

The quantity $\sum m_ir_i^2$ is called the object's **moment of inertia** I:

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots = \sum_i m_ir_i^2 \qquad (12.12)$$

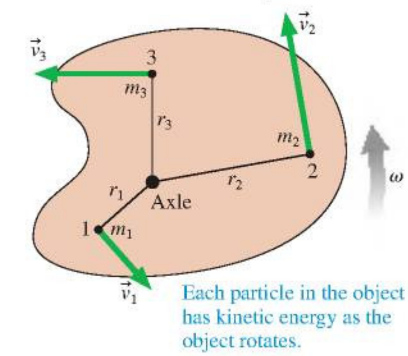
The units of moment of inertia are kg m^2. **An object's moment of inertia depends on the axis of rotation.** Once the axis is specified, allowing the values of r_i to be determined, the moment of inertia *about that axis* can be calculated from Equation 12.12.

Written using the moment of inertia I, the rotational kinetic energy is

$$K_{rot} = \tfrac{1}{2}I\omega^2 \qquad (12.13)$$

NOTE Rotational kinetic energy is *not* a new form of energy. It is the familiar kinetic energy of motion, simply expressed in a form that is convenient for rotational motion. Notice the analogy with the familiar $\tfrac{1}{2}mv^2$.

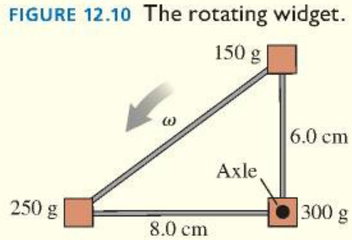
FIGURE 12.9 Rotational kinetic energy is due to the motion of the particles.



EXAMPLE 12.3 **A rotating widget**

Students participating in an engineering project design the triangular widget seen in FIGURE 12.10. The three masses, held together by lightweight plastic rods, rotate in the plane of the page about an axle passing through the right-angle corner. At what angular velocity does the widget have 100 mJ of rotational energy?

FIGURE 12.10 The rotating widget.



MODEL Model the widget as a rigid body consisting of three particles connected by massless rods.

SOLVE Rotational energy is $K = \tfrac{1}{2}I\omega^2$. The moment of inertia is measured about the rotation axis, thus

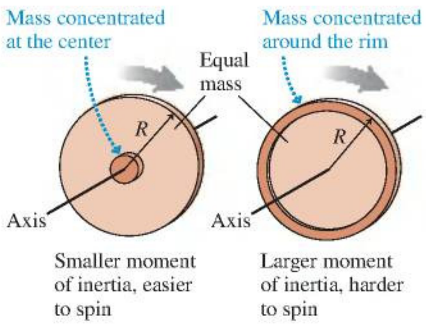
$$I = \sum_i m_ir_i^2 = (0.25\text{ kg})(0.080\text{ m})^2 + (0.15\text{ kg})(0.060\text{ m})^2 + (0.30\text{ kg})(0\text{ m})^2$$
$$= 2.14 \times 10^{-3}\text{ kg m}^2$$

The largest mass makes no contribution to I because it is on the rotation axis with $r = 0$. With I known, the angular velocity is

$$\omega = \sqrt{\frac{2K}{I}} = \sqrt{\frac{2(0.10\text{ J})}{2.14 \times 10^{-3}\text{ kg m}^2}}$$
$$= 9.67\text{ rad/s} \times \frac{1\text{ rev}}{2\pi\text{ rad}} = 1.54\text{ rev/s} = 92\text{ rpm}$$

ASSESS The moment of inertia depends on the distance of each mass from the rotation axis. The moment of inertia would be different for an axle passing through either of the other two masses, and thus the required angular velocity would be different.

FIGURE 12.11 Moment of inertia depends on both the mass and how the mass is distributed.

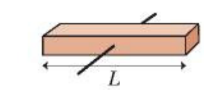
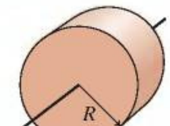
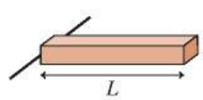
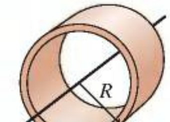
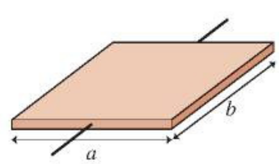
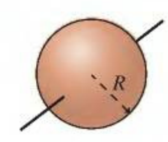
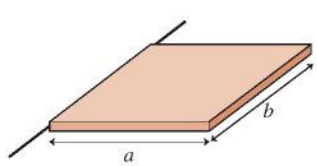
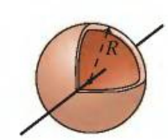


Before rushing to calculate moments of inertia, let's get a better understanding of the meaning. First, notice that **moment of inertia is the rotational equivalent of mass**. It plays the same role in Equation 12.13 as mass m in the now-familiar $K = \frac{1}{2}mv^2$. Recall that the quantity we call *mass* was actually defined as the *inertial mass*. Objects with larger mass have a larger *inertia*, meaning that they're harder to accelerate. Similarly, an object with a larger moment of inertia is harder to rotate. The fact that *moment of inertia* retains the word "inertia" reminds us of this.

Consider the two wheels shown in **FIGURE 12.11**. They have the same total mass M and the same radius R. As you probably know from experience, it's much easier to spin the wheel whose mass is concentrated at the center than to spin the one whose mass is concentrated around the rim. This is because having the mass near the center (smaller values of r_i) lowers the moment of inertia.

Moments of inertia for many solid objects are tabulated and found online. You would need to compute I yourself only for an object of unusual shape. **TABLE 12.2** is a short list of common moments of inertia. We'll see in the next section where these come from, but do notice how I depends on the rotation axis.

TABLE 12.2 Moments of inertia of objects with uniform density

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

If the rotation axis is not through the center of mass, then rotation may cause the center of mass to move up or down. In that case, the object's gravitational potential energy $U_G = Mgy_{cm}$ will change. If there are no dissipative forces (i.e., if the axle is frictionless) and if no work is done by external forces, then the mechanical energy

$$E_{mech} = K_{rot} + U_G = \frac{1}{2}I\omega^2 + Mgy_{cm} \qquad (12.14)$$

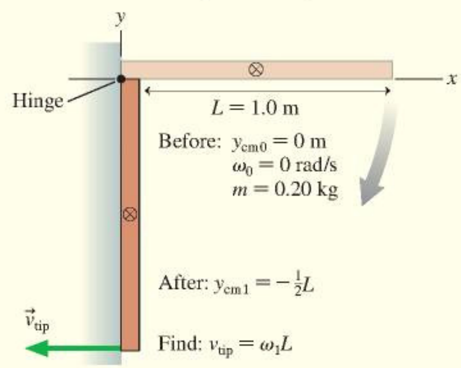
is a conserved quantity.

EXAMPLE 12.4 **The speed of a rotating rod**

A 1.0-m-long, 200 g rod is hinged at one end and connected to a wall. It is held out horizontally, then released. What is the speed of the tip of the rod as it hits the wall?

MODEL The mechanical energy is conserved if we assume the hinge is frictionless. The rod's gravitational potential energy is transformed into rotational kinetic energy as it "falls."

FIGURE 12.12 A before-and-after pictorial representation of the rod.



VISUALIZE FIGURE 12.12 is a familiar before-and-after pictorial representation of the rod.

SOLVE Mechanical energy is conserved, so we can equate the rod's final mechanical energy to its initial mechanical energy:

$$\frac{1}{2}I\omega_1^2 + Mgy_{cm1} = \frac{1}{2}I\omega_0^2 + Mgy_{cm0}$$

The initial conditions are $\omega_0 = 0$ and $y_{cm0} = 0$. The center of mass moves to $y_{cm1} = -\frac{1}{2}L$ as the rod hits the wall. From Table 12.2 we find $I = \frac{1}{3}ML^2$ for a rod rotating about one end. Thus

$$\frac{1}{2}I\omega_1^2 + Mgy_{cm1} = \frac{1}{6}ML^2\omega_1^2 - \frac{1}{2}MgL = 0$$

We can solve this for the rod's angular velocity as it hits the wall:

$$\omega_1 = \sqrt{\frac{3g}{L}}$$

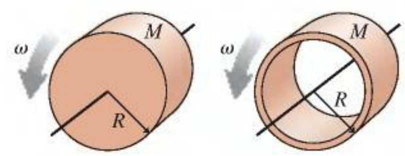
The tip of the rod is moving in a circle with radius $r = L$. Its final speed is

$$v_{tip} = \omega_1 L = \sqrt{3gL} = 5.4 \text{ m/s}$$

ASSESS 5.4 m/s = 12 mph, which seems plausible for a meter-long rod swinging through 90°.

STOP TO THINK 12.2 A solid cylinder and a cylindrical shell, each with radius R and mass M, rotate about their axes with the same angular velocity ω. Which has more kinetic energy?

a. The solid cylinder.
b. The cylindrical shell.
c. They have the same kinetic energy.
d. Neither has kinetic energy because they are only rotating, not moving.



12.4 Calculating Moment of Inertia

The equation for rotational energy is easy to write, but we can't make use of it without knowing an object's moment of inertia. Unlike mass, we can't measure moment of inertia by putting an object on a scale. And while we can guess that the center of mass of a symmetrical object is at the physical center of the object, we can *not* guess the moment of inertia of even a simple object. To find I, we really must carry through the calculation.

Equation 12.12 defines the moment of inertia as a sum over all the particles in the system. As we did for the center of mass, we can replace the individual particles with cells 1, 2, 3, … of mass Δm. Then the moment of inertia summation can be converted to an integration:

$$I = \sum_i r_i^2 \,\Delta m \xrightarrow[\Delta m \to 0]{} I = \int r^2 \, dm \qquad (12.15)$$

where r is the distance from the rotation axis. If we let the rotation axis be the z-axis, then we can write the moment of inertia as

$$I = \int (x^2 + y^2)\, dm \qquad \text{(rotation about the } z\text{-axis)} \qquad (12.16)$$

NOTE You *must* replace dm by an equivalent expression involving a coordinate differential such as dx or dy before you can carry out the integration.

You can use any coordinate system to calculate the coordinates x_{cm} and y_{cm} of the center of mass. But the moment of inertia is defined for rotation about a particular axis, and r is measured from that axis. Thus the coordinate system used for moment-of-inertia calculations *must* have its origin at the pivot point.

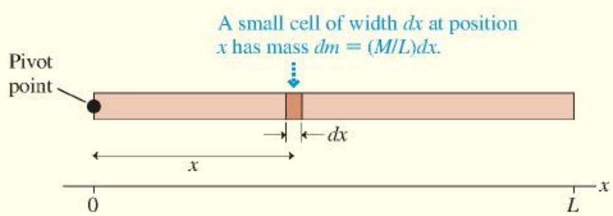
EXAMPLE 12.5 Moment of inertia of a rod about a pivot at one end

Find the moment of inertia of a thin, uniform rod of length L and mass M that rotates about a pivot at one end.

MODEL An object's moment of inertia depends on the axis of rotation. In this case, the rotation axis is at the end of the rod.

VISUALIZE FIGURE 12.13 defines an x-axis with the origin at the pivot point.

FIGURE 12.13 Setting up the integral to find the moment of inertia of a rod.



SOLVE Because the rod is thin, we can assume that $y \approx 0$ for all points on the rod. Thus

$$I = \int x^2\, dm$$

The small amount of mass dm in the small length dx is $dm = (M/L)\,dx$, as we found in Example 12.2. The rod extends from $x = 0$ to $x = L$, so the moment of inertia about one end is

$$I_{end} = \frac{M}{L}\int_0^L x^2\, dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \tfrac{1}{3}ML^2$$

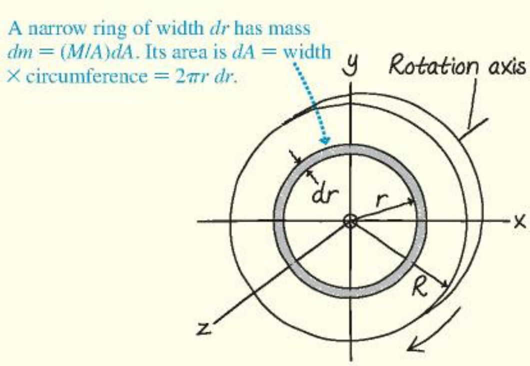
ASSESS The moment of inertia involves a product of the total mass M with the *square* of a length, in this case L. All moments of inertia have a similar form, although the fraction in front will vary. This is the result shown earlier in Table 12.2.

EXAMPLE 12.6 Moment of inertia of a circular disk about an axis through the center

Find the moment of inertia of a circular disk of radius R and mass M that rotates on an axis passing through its center.

VISUALIZE FIGURE 12.14 shows the disk and defines distance r from the axis.

FIGURE 12.14 Setting up the integral to find the moment of inertia of a disk.



SOLVE This is a situation of great practical importance. To solve this problem, we need to use a two-dimensional integration scheme that you learned in calculus. Rather than dividing the disk into little boxes, let's divide it into narrow *rings* of mass dm. Figure 12.14 shows one such ring, of radius r and width dr. Let dA represent the area of this ring. The mass dm in this ring is the same fraction of the total mass M as dA is of the total area A. That is,

$$\frac{dm}{M} = \frac{dA}{A}$$

Thus the mass in the small area dA is

$$dm = \frac{M}{A}\, dA$$

This is the reasoning we used to find the center of mass of the rod in Example 12.2, only now we're using it in two dimensions.

The total area of the disk is $A = \pi R^2$, but what is dA? If we imagine unrolling the little ring, it would form a long, thin rectangle of length $2\pi r$ and height dr. Thus the *area* of this little ring is $dA = 2\pi r\, dr$. With this information we can write

$$dm = \frac{M}{\pi R^2}(2\pi r\, dr) = \frac{2M}{R^2} r\, dr$$

Now we have an expression for dm in terms of a coordinate differential dr, so we can proceed to carry out the integration for I. Using Equation 12.15, we find

$$I_{disk} = \int r^2\, dm = \int r^2\left(\frac{2M}{R^2} r\, dr\right) = \frac{2M}{R^2}\int_0^R r^3\, dr$$

where in the last step we have used the fact that the disk extends from $r = 0$ to $r = R$. Performing the integration gives

$$I_{disk} = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \tfrac{1}{2}MR^2$$

ASSESS Once again, the moment of inertia involves a product of the total mass M with the *square* of a length, in this case R.

If a complex object can be divided into simpler pieces 1, 2, 3, . . . whose moments of inertia $I_1, I_2, I_3 \ldots$ are already known, the moment of inertia of the entire object is

$$I_{object} = I_1 + I_2 + I_3 + \cdots \qquad (12.17)$$

The Parallel-Axis Theorem

The moment of inertia depends on the rotation axis. Suppose you need to know the moment of inertia for rotation about the off-center axis in **FIGURE 12.15**. You can find this quite easily if you know the moment of inertia for rotation around a *parallel axis* through the center of mass.

If the axis of interest is distance d from a parallel axis through the center of mass, the moment of inertia is

$$I = I_{cm} + Md^2 \tag{12.18}$$

Equation 12.18 is called the **parallel-axis theorem.** We'll give a proof for the one-dimensional object shown in **FIGURE 12.16**.

The x-axis has its origin at the rotation axis, and the x'-axis has its origin at the center of mass. You can see that the coordinates of dm along these two axes are related by $x = x' + d$. By definition, the moment of inertia about the rotation axis is

$$I = \int x^2\, dm = \int (x' + d)^2\, dm = \int (x')^2\, dm + 2d \int x'\, dm + d^2 \int dm \tag{12.19}$$

The first of the three integrals on the right, by definition, is the moment of inertia I_{cm} about the center of mass. The third is simply Md^2 because adding up (integrating) all the dm gives the total mass M.

If you refer back to Equations 12.5, the definition of the center of mass, you'll see that the middle integral on the right is equal to Mx'_{cm}. But $x'_{cm} = 0$ because we specifically chose the x'-axis to have its origin at the center of mass. Thus the second integral is zero and we end up with Equation 12.18. The proof in two dimensions is similar.

FIGURE 12.15 An off-center axis.

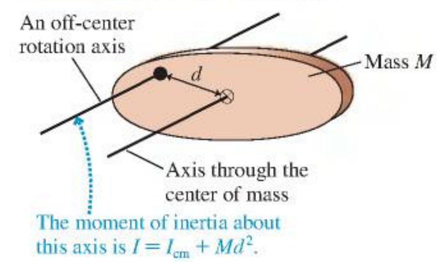
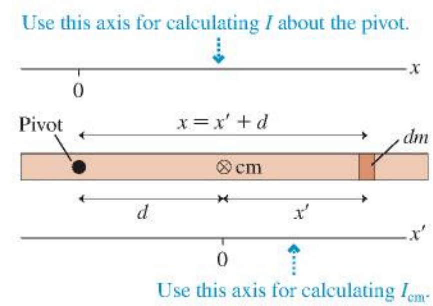


FIGURE 12.16 Proving the parallel-axis theorem.



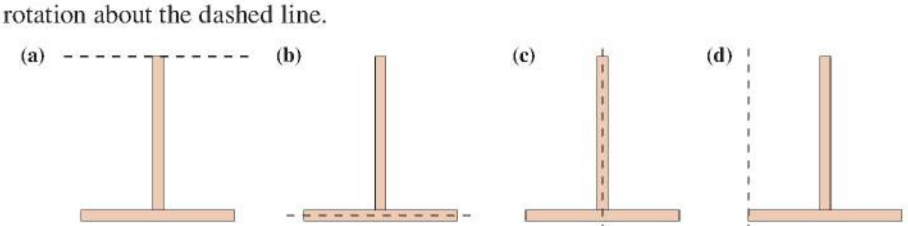
EXAMPLE 12.7 **The moment of inertia of a thin rod**

Find the moment of inertia of a thin rod with mass M and length L about an axis one-third of the length from one end.

SOLVE From Table 12.2 we know the moment of inertia about the center of mass is $\frac{1}{12}ML^2$. The center of mass is at the center of the rod. An axis $\frac{1}{3}L$ from one end is $d = \frac{1}{6}L$ from the center of mass. Using the parallel-axis theorem, we have

$$I = I_{cm} + Md^2 = \tfrac{1}{12}ML^2 + M\left(\tfrac{1}{6}L\right)^2 = \tfrac{1}{9}ML^2$$

STOP TO THINK 12.3 Four Ts are made from two identical rods of equal mass and length. Rank in order, from largest to smallest, the moments of inertia I_a to I_d for rotation about the dashed line.

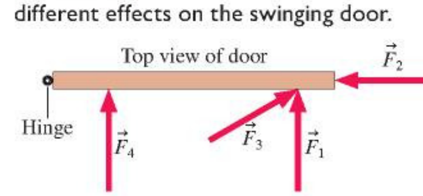


12.5 Torque

Consider the common experience of pushing open a door. **FIGURE 12.17** is a top view of a door hinged on the left. Four pushing forces are shown, all of equal strength. Which of these will be most effective at opening the door?

Force $\vec{F}_1$ will open the door, but force $\vec{F}_2$, which pushes straight at the hinge, will not. Force $\vec{F}_3$ will open the door, but not as easily as $\vec{F}_1$. What about $\vec{F}_4$? It is perpendicular to the door, it has the same magnitude as $\vec{F}_1$, but you know from

FIGURE 12.17 The four forces have different effects on the swinging door.



experience that pushing close to the hinge is not as effective as pushing at the outer edge of the door.

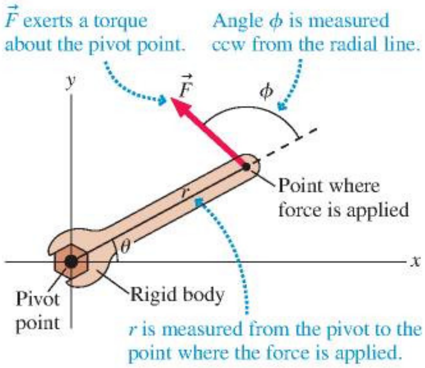
The ability of a force to cause a rotation depends on three factors:

1. The magnitude F of the force.
2. The distance r from the point of application to the pivot.
3. The angle at which the force is applied.

We can incorporate these three factors into a single quantity called the *torque*. FIGURE 12.18 shows a force $\vec{F}$ trying to rotate the wrench and nut about a *pivot point*—the axis about which the nut will rotate. We say that this force exerts a **torque** τ (Greek tau), which we define as

$$\tau \equiv rF\sin\phi \tag{12.20}$$

FIGURE 12.18 Force $\vec{F}$ exerts a torque about the pivot point.



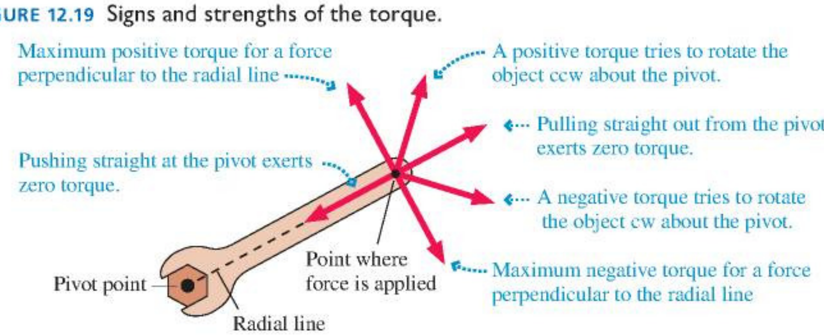
Torque depends on the three properties we just listed: the magnitude of the force, its distance from the pivot, and its angle. Loosely speaking, τ measures the "effectiveness" of the force at causing an object to rotate about a pivot. **Torque is the rotational equivalent of force.**

NOTE Angle ϕ is measured *counterclockwise* from the dashed line that extends outward along the radial line. This is consistent with our sign convention for the angular position θ.

The SI units of torque are newton-meters, abbreviated N m. Although we defined 1 N m = 1 J during our study of energy, torque is not an energy-related quantity and so we do *not* use joules as a measure of torque.

Torque, like force, has a sign. A torque that tries to rotate the object in a ccw direction is positive while a negative torque gives a cw rotation. FIGURE 12.19 summarizes the signs. Notice that a force pushing straight toward the pivot or pulling straight out from the pivot exerts *no* torque.

FIGURE 12.19 Signs and strengths of the torque.



Your foot exerts a torque that rotates the crank.

NOTE Torque differs from force in a very important way. Torque is calculated or measured *about a pivot point*. To say that a torque is 20 N m is meaningless. You need to say that the torque is 20 N m about a particular point. Torque can be calculated about any pivot point, but its value depends on the point chosen because this choice determines r and ϕ.

Returning to the door of Figure 12.17, you can see that $\vec{F}_1$ is most effective at opening the door because $\vec{F}_1$ exerts the largest torque *about the pivot point*. $\vec{F}_3$ has equal magnitude, but it is applied at an angle less than 90° and thus exerts less torque. $\vec{F}_2$, pushing straight at the hinge with $\phi = 0°$, exerts no torque at all. And $\vec{F}_4$, with a smaller value for r, exerts less torque than $\vec{F}_1$.

Interpreting Torque

Torque can be interpreted from two perspectives, as shown in **FIGURE 12.20**. First, the quantity $F\sin\phi$ is the tangential force component F_t. Consequently, the torque is

$$\tau = rF_t \tag{12.21}$$

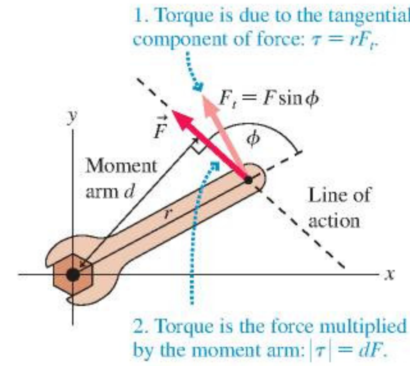
In other words, torque is the product of r with the force component F_t that is tangent to the circular path followed by this point on the wrench. This interpretation makes sense because the radial component of $\vec{F}$ points straight at the pivot point and cannot exert a torque.

A second perspective, widely used in applications, is based on the idea of a *moment arm*. Figure 12.20 shows the **line of action,** the line along which the force acts. The minimum distance between the pivot point and the line of action—the length of a line drawn *perpendicular to the line of action*—is called the **moment arm** (or the *lever arm*) d. Because $\sin(180° - \phi) = \sin\phi$, it is easy to see that $d = r\sin\phi$. Thus the torque $rF\sin\phi$ can also be written

$$|\tau| = dF \tag{12.22}$$

NOTE Equation 12.22 gives only $|\tau|$, the magnitude of the torque; the sign has to be supplied by observing the direction in which the torque acts.

FIGURE 12.20 Two useful interpretations of the torque.

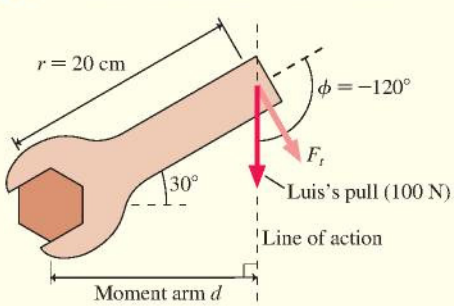


EXAMPLE 12.8 Applying a torque

Luis uses a 20-cm-long wrench to turn a nut. The wrench handle is tilted 30° above the horizontal, and Luis pulls straight down on the end with a force of 100 N. How much torque does Luis exert on the nut?

VISUALIZE **FIGURE 12.21** shows the situation. The angle is a negative $\phi = -120°$ because it is *clockwise* from the radial line.

FIGURE 12.21 A wrench being used to turn a nut.



SOLVE The tangential component of the force is

$$F_t = F\sin\phi = -86.6\ \text{N}$$

According to our sign convention, F_t is negative because it points in a cw direction. The torque, from Equation 12.21, is

$$\tau = rF_t = (0.20\ \text{m})(-86.6\ \text{N}) = -17\ \text{N m}$$

Alternatively, Figure 12.21 has drawn the *line of action* by extending the force vector forward and backward. The *moment arm*, the distance between the pivot point and the line of action, is

$$d = r\sin(60°) = 0.17\ \text{m}$$

Inserting the moment arm in Equation 12.22 gives

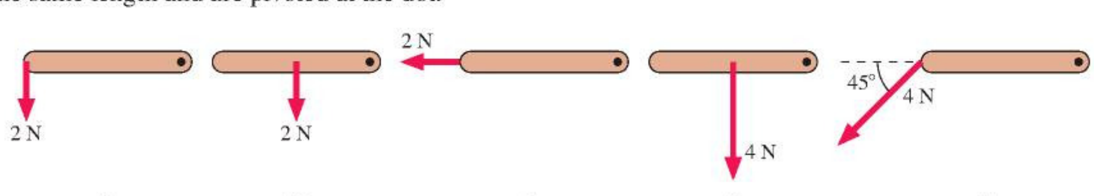
$$|\tau| = dF = (0.17\ \text{m})(100\ \text{N}) = 17\ \text{N m}$$

The torque acts to give a cw rotation, so we insert a minus sign to end up with

$$\tau = -17\ \text{N m}$$

ASSESS The largest possible torque, if Luis pulled perpendicular to the 20-cm-long wrench, would have a magnitude of 20 N m. Pulling at an angle will reduce this, so 17 N m is a reasonable answer.

STOP TO THINK 12.4 Rank in order, from largest to smallest, the five torques τ_a to τ_e. The rods all have the same length and are pivoted at the dot.



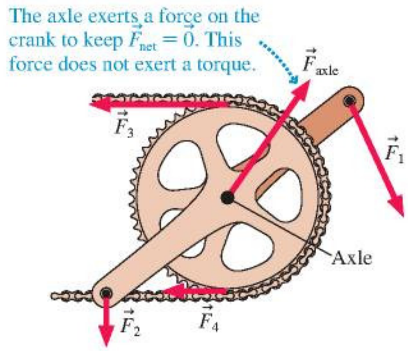


FIGURE 12.22 The forces exert a net torque about the pivot point.

Net Torque

FIGURE 12.22 shows the forces acting on the crankset of a bicycle. The crankset is free to rotate about the axle, but the axle prevents it from having any translational motion relative to the bike frame. It does so by exerting force $\vec{F}_{axle}$ on the crankset to balance the other forces and keep $\vec{F}_{net} = \vec{0}$.

Forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots$ exert torques $\tau_1, \tau_2, \tau_3, \ldots$ on the crankset, but $\vec{F}_{axle}$ does *not* exert a torque because it is applied at the pivot point and has zero moment arm. Thus the *net* torque about the axle is the sum of the torques due to the applied forces:

$$\tau_{net} = \tau_1 + \tau_2 + \tau_3 + \cdots = \sum_i \tau_i \tag{12.23}$$

Gravitational Torque

Gravity exerts a torque on many objects. If the object in **FIGURE 12.23** is released, a torque due to gravity will cause it to rotate around the axle. To calculate the torque about the axle, we start with the fact that gravity acts on *every* particle in the object, exerting a downward force of magnitude $F_i = m_i g$ on particle i. The *magnitude* of the gravitational torque on particle i is $|\tau_i| = d_i m_i g$, where d_i is the moment arm. But we need to be careful with signs.

A moment arm must be a positive number because it's a distance. If we establish a coordinate system with the origin at the axle, then you can see from Figure 12.23a that the moment arm d_i of particle i is $|x_i|$. A particle to the right of the axle (positive x_i) experiences a *negative* torque because gravity tries to rotate this particle in a clockwise direction. Similarly, a particle to the left of the axle (negative x_i) has a positive torque. The torque is opposite in sign to x_i, so we can get the sign right by writing

$$\tau_i = -x_i m_i g = -(m_i x_i)g \tag{12.24}$$

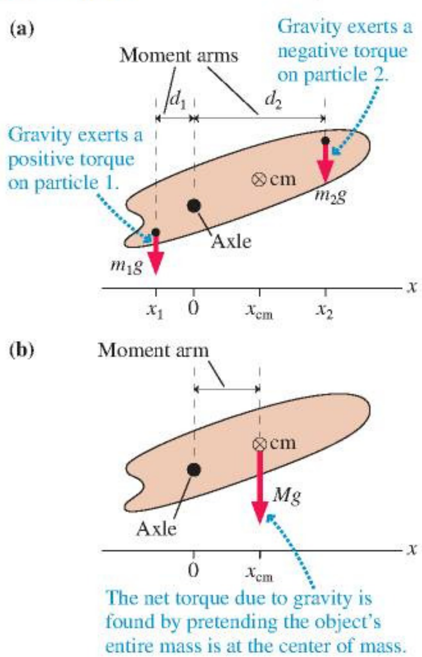


FIGURE 12.23 Gravitational torque.

The net torque due to gravity is found by summing Equation 12.24 over all particles:

$$\tau_{grav} = \sum_i \tau_i = \sum_i (-m_i x_i g) = -\left(\sum_i m_i x_i\right) g \tag{12.25}$$

But according to the definition of center of mass, Equations 12.4, $\sum m_i x_i = Mx_{cm}$. Thus the torque due to gravity is

$$\tau_{grav} = -Mgx_{cm} \tag{12.26}$$

where x_{cm} is the position of the center of mass *relative to the axis of rotation.*

Equation 12.26 has the simple interpretation shown in Figure 12.23b. Mg is the net gravitational force on the entire object, and x_{cm} is the moment arm between the rotation axis and the center of mass. The gravitational torque on an extended object of mass M is equivalent to the torque of a *single* force vector $\vec{F}_G = -Mg\hat{\jmath}$ acting at the object's center of mass.

In other words, **the gravitational torque is found by treating the object as if all its mass were concentrated at the center of mass.** This is the basis for the well-known technique of finding an object's center of mass by balancing it. An object will balance on a pivot, as shown in **FIGURE 12.24**, only if the center of mass is directly above the pivot point. If the pivot is *not* under the center of mass, the gravitational torque will cause the object to rotate.

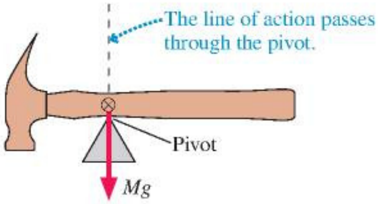


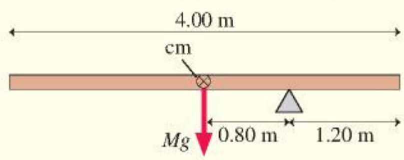
FIGURE 12.24 An object balances on a pivot that is directly under the center of mass.

NOTE The point at which gravity acts is also called the *center of gravity.* As long as gravity is uniform over the object—always true for earthbound objects—there's no difference between center of mass and center of gravity.

EXAMPLE 12.9 **The gravitational torque on a beam**

The 4.00-m-long, 500 kg steel beam shown in FIGURE 12.25 is supported 1.20 m from the right end. What is the gravitational torque about the support?

FIGURE 12.25 A steel beam supported at one point.



MODEL The center of mass of the beam is at the midpoint. $x_{cm} = -0.80$ m is measured from the pivot point.

SOLVE This is a straightforward application of Equation 12.26. The gravitational torque is

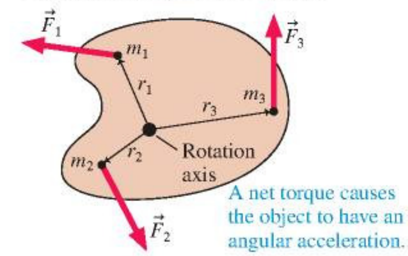
$$\tau_{grav} = -Mgx_{cm} = -(500 \text{ kg})(9.80 \text{ m/s}^2)(-0.80 \text{ m}) = 3920 \text{ N m}$$

ASSESS The torque is positive because gravity tries to rotate the beam ccw. Notice that the beam in Figure 12.25 is *not* in equilibrium. It will fall over unless other forces, not shown, are supporting it.

12.6 Rotational Dynamics

What does a torque do? **A torque causes an angular acceleration.** To see why, FIGURE 12.26 shows a rigid body undergoing *pure rotational motion* about a fixed and unmoving axis. This might be a rotation about the object's center of mass, such as we considered in Section 12.2. Or it might be an object, such as a turbine, rotating on an axle.

FIGURE 12.26 The external forces exert a torque about the rotation axis.



The forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots$ in Figure 12.26 are external forces acting on particles of masses $m_1, m_2, m_3, \ldots$ that are part of the rigid body. These forces exert torques τ_1, τ_2, $\tau_3, \ldots$ about the rotation axis. The *net* torque on the object is the sum of the torques:

$$\tau_{net} = \sum_i \tau_i \tag{12.27}$$

Focus on particle i, which is acted on by force $\vec{F}_i$ and undergoes circular motion with radius r_i. In Chapter 8, we found that the radial component of $\vec{F}_i$ is responsible for the centripetal acceleration of circular motion, while the tangential component $(F_i)_t$ causes the particle to speed up or slow down with a tangential acceleration $(a_i)_t$. Newton's second law is

$$(F_i)_t = m_i(a_i)_t = m_i r_i \alpha \tag{12.28}$$

where in the last step we used the relationship between tangential and angular acceleration: $a_t = r\alpha$. The angular acceleration α does not have a subscript because *all particles in the object have the same angular acceleration.* That is, α is the angular acceleration of the entire object.

Multiplying both sides by r_i gives

$$r_i(F_i)_t = m_i r_i^2 \alpha \tag{12.29}$$

But $r_i(F_i)_t$ is the torque τ_i about the axis on particle i; hence Newton's second law for a single particle in the object is

$$\tau_i = m_i r_i^2 \alpha \tag{12.30}$$

Returning now to Equation 12.27, we see that the net torque on the object is

$$\tau_{net} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha = \left(\sum_i m_i r_i^2\right)\alpha \tag{12.31}$$

In the last step, we factored out α by using the key idea that every particle in a rotating rigid body has the *same* angular acceleration.

You'll recognize the quantity in parentheses as the moment of inertia. Substituting I into Equation 12.31 puts the final piece of the puzzle into place. An object that experiences a net torque τ_{net} about the axis of rotation undergoes an angular acceleration

$$\alpha = \frac{\tau_{net}}{I} \quad \text{(Newton's second law for rotational motion)} \tag{12.32}$$

where I is the object's moment of inertia *about the rotation axis.* This result, Newton's second law for rotation, is the fundamental equation of rigid-body dynamics.

In practice we often write $\tau_{net} = I\alpha$, but Equation 12.32 better conveys the idea that **torque is the cause of angular acceleration.** In the absence of a net torque ($\tau_{net} = 0$), the object either does not rotate ($\omega = 0$) or rotates with *constant* angular velocity (ω = constant).

TABLE 12.3 summarizes the analogies between linear and rotational dynamics.

TABLE 12.3 Rotational and linear dynamics

Rotational dynamics		Linear dynamics	
torque (N m)	τ_{net}	force (N)	$\vec{F}_{net}$
moment of inertia (kg m^2)	I	mass (kg)	m
angular acceleration (rad/s^2)	α	acceleration (m/s^2)	$\vec{a}$
second law	$\alpha = \tau_{net}/I$	second law	$\vec{a} = \vec{F}_{net}/m$

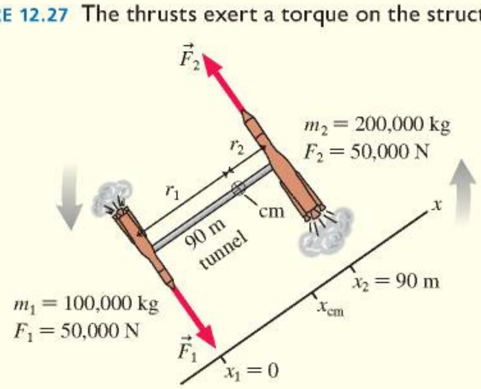
EXAMPLE 12.10 Rotating rockets

Far out in space, a 100,000 kg rocket and a 200,000 kg rocket are docked at opposite ends of a motionless 90-m-long connecting tunnel. The tunnel is rigid and its mass is much less than that of either rocket. The rockets start their engines simultaneously, each generating 50,000 N of thrust in opposite directions. What is the structure's angular velocity after 30 s?

MODEL The entire structure can be modeled as two masses at the ends of a massless, rigid rod. There's no net force, so the structure does not undergo translational motion, but the thrusts do create torques that will give the structure angular acceleration and cause it to rotate. We'll assume the thrust forces are perpendicular to the connecting tunnel. This is an unconstrained rotation, so the structure will rotate about its center of mass.

VISUALIZE FIGURE 12.27 shows the rockets and defines distances r_1 and r_2 from the center of mass.

FIGURE 12.27 The thrusts exert a torque on the structure.



SOLVE Our strategy will be to use Newton's second law to find the angular acceleration, followed by rotational kinematics to find ω. We'll need to determine the moment of inertia, and that requires knowing the distances of the two rockets from the rotation axis. As we did in Example 12.1, we choose a coordinate system in which the masses are on the x-axis and in which m_1 is at the origin. Then

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$
$$= \frac{(100{,}000\text{ kg})(0\text{ m}) + (200{,}000\text{ kg})(90\text{ m})}{100{,}000\text{ kg} + 200{,}000\text{ kg}} = 60\text{ m}$$

The structure's center of mass is $r_1 = 60$ m from the 100,000 kg rocket and $r_2 = 30$ m from the 200,000 kg rocket. The moment of inertia about the center of mass is

$$I = m_1 r_1^2 + m_2 r_2^2 = 540{,}000{,}000\text{ kg m}^2$$

The two rocket thrusts exert net torque

$$\tau_{net} = r_1 F_1 + r_2 F_2 = (60\text{ m})(50{,}000\text{ N}) + (30\text{ m})(50{,}000\text{ N})$$
$$= 4{,}500{,}000\text{ N m}$$

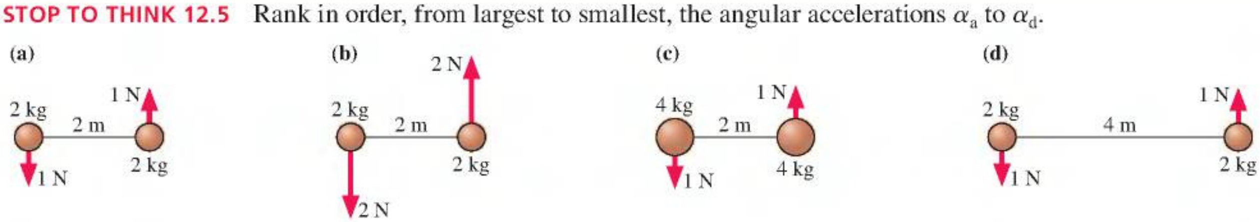
With I and τ_{net} now known, we can use Newton's second law to find the angular acceleration:

$$\alpha = \frac{\tau}{I} = \frac{4{,}500{,}000\text{ N m}}{540{,}000{,}000\text{ kg m}^2} = 0.00833\text{ rad/s}^2$$

After 30 seconds, the structure's angular velocity is

$$\omega = \alpha\,\Delta t = 0.25\text{ rad/s}$$

STOP TO THINK 12.5 Rank in order, from largest to smallest, the angular accelerations α_a to α_d.



12.7 Rotation About a Fixed Axis

In this section we'll look at rigid bodies that rotate about a fixed axis. The problem-solving strategy for rotational dynamics is very similar to that for linear dynamics.

PROBLEM-SOLVING STRATEGY 12.1

Rotational dynamics problems

MODEL Model the object as a rigid body.

VISUALIZE Draw a pictorial representation to clarify the situation, define coordinates and symbols, and list known information.

- Identify the axis about which the object rotates.
- Identify forces and determine their distances from the axis. For most problems it will be useful to draw a free-body diagram.
- Identify any torques caused by the forces and the signs of the torques.

SOLVE The mathematical representation is based on Newton's second law for rotational motion:

$$\tau_{net} = I\alpha \qquad \text{or} \qquad \alpha = \frac{\tau_{net}}{I}$$

- Find the moment of inertia in Table 12.2 or, if needed, calculate it as an integral or by using the parallel-axis theorem.
- Use rotational kinematics to find angles and angular velocities.

ASSESS Check that your result has correct units and significant figures, is reasonable, and answers the question.

Exercise 28

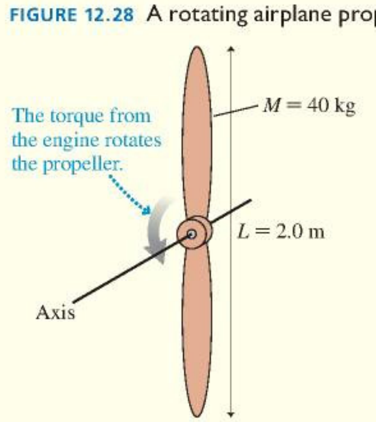
EXAMPLE 12.11 | **Starting an airplane engine**

The engine in a small airplane is specified to have a torque of 60 N m. This engine drives a 2.0-m-long, 40 kg propeller. On start-up, how long does it take the propeller to reach 200 rpm?

MODEL The propeller can be modeled as a rigid rod that rotates about its center. The engine exerts a torque on the propeller.

VISUALIZE **FIGURE 12.28** shows the propeller and the rotation axis.

FIGURE 12.28 A rotating airplane propeller.



SOLVE The moment of inertia of a rod rotating about its center is found from Table 12.2:

$$I = \tfrac{1}{12}ML^2 = \tfrac{1}{12}(40\text{ kg})(2.0\text{ m})^2 = 13.33\text{ kg m}^2$$

The 60 N m torque of the engine causes an angular acceleration

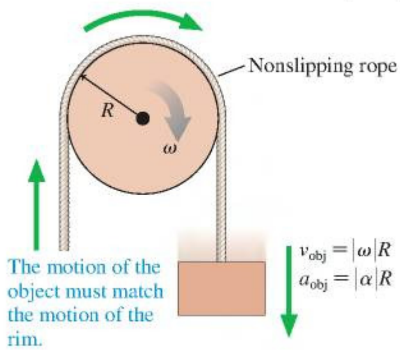
$$\alpha = \frac{\tau}{I} = \frac{60\text{ N m}}{13.33\text{ kg m}^2} = 4.50\text{ rad/s}^2$$

The time needed to reach $\omega_f = 200\text{ rpm} = 3.33\text{ rev/s} = 20.9\text{ rad/s}$ is

$$\Delta t = \frac{\Delta\omega}{\alpha} = \frac{\omega_f - \omega_i}{\alpha} = \frac{20.9\text{ rad/s} - 0\text{ rad/s}}{4.5\text{ rad/s}^2} = 4.6\text{ s}$$

ASSESS We've assumed a constant angular acceleration, which is reasonable for the first few seconds while the propeller is still turning slowly. Eventually, air resistance and friction will cause opposing torques and the angular acceleration will decrease. At full speed, the negative torque due to air resistance and friction cancels the torque of the engine. Then $\tau_{net} = 0$ and the propeller turns at *constant* angular velocity with no angular acceleration.

FIGURE 12.29 The rope's motion must match the motion of the rim of the pulley.



Constraints Due to Ropes and Pulleys

Many important applications of rotational dynamics involve objects, such as pulleys, that are connected via ropes or belts to other objects. FIGURE 12.29 shows a rope passing over a pulley and connected to an object in linear motion. If the rope does not slip as the pulley rotates, then the rope's speed v_{rope} must exactly match the speed of the rim of the pulley, which is $v_{rim} = |\omega|R$. If the pulley has an angular acceleration, the rope's acceleration a_{rope} must match the *tangential* acceleration of the rim of the pulley, $a_t = |\alpha|R$.

The object attached to the other end of the rope has the same speed and acceleration as the rope. Consequently, an object connected to a pulley of radius R by a rope that does not slip must obey the constraints

$$\left.\begin{aligned} v_{obj} &= |\omega|R \\ a_{obj} &= |\alpha|R \end{aligned}\right\} \quad \text{(motion constraints for a nonslipping rope)} \qquad (12.33)$$

These constraints are very similar to the acceleration constraints introduced in Chapter 7 for two objects connected by a string or rope.

NOTE The constraints are given as magnitudes. Specific problems will need to introduce signs that depend on the direction of motion and on the choice of coordinate system.

The Constant-Torque Model

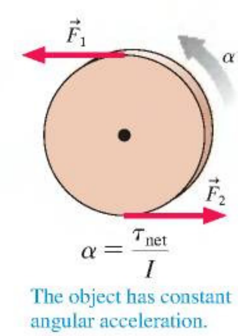
If all the torques exerted on an object are constant, the object rotates with constant angular acceleration. Even if the torques aren't perfectly constant, there are many situations where it's reasonable to model them as if they were. The **constant-torque model,** analogous to the constant-force model of Section 6.2, is the most important model of rotational dynamics.

MODEL 12.2

Constant torque

For objects on which the net torque is constant.

- Model the object as a rigid body with constant angular acceleration.
- Take into account constraints due to ropes and pulleys.
- Mathematically:
  - **Newton's second law** is $\tau_{net} = I\alpha$.
  - Use the kinematics of constant angular acceleration.
- Limitations: Model fails if the torque isn't constant.



EXAMPLE 12.12 Lowering a bucket

A 2.0 kg bucket is attached to a massless string that is wrapped around a 1.0 kg, 4.0-cm-diameter cylinder, as shown in FIGURE 12.30a. The cylinder rotates on an axle through the center. The bucket is released from rest 1.0 m above the floor. How long does it take to reach the floor?

MODEL Assume the string does not slip.

VISUALIZE FIGURE 12.30b shows the free-body diagram for the cylinder and the bucket. The string tension exerts an upward force on the bucket and a downward force on the outer edge of the cylinder. The string is massless, so these two tension forces act as if they are an action/reaction pair: $T_b = T_c = T$.

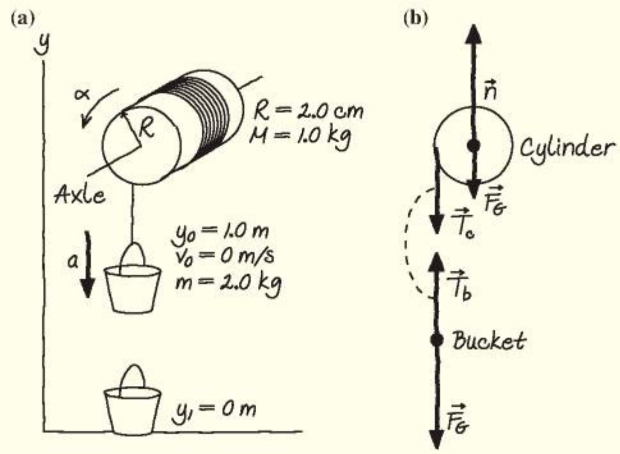
SOLVE Newton's second law applied to the linear motion of the bucket is

$$ma_y = T - mg$$

where, as usual, the y-axis points upward. What about the cylinder? The only torque comes from the string tension. The moment arm for the tension is $d = R$, and the torque is positive because the string turns the cylinder ccw. Thus $\tau_{string} = TR$ and Newton's second law for the rotational motion is

$$\alpha = \frac{\tau_{net}}{I} = \frac{TR}{\frac{1}{2}MR^2} = \frac{2T}{MR}$$

FIGURE 12.30 The falling bucket turns the cylinder.



The moment of inertia of a cylinder rotating about a center axis was taken from Table 12.2.

The last piece of information we need is the constraint due to the fact that the string doesn't slip. Equation 12.33 relates only the absolute values, but in this problem α is positive (ccw acceleration) while a_y is negative (downward acceleration). Hence

$$a_y = -\alpha R$$

Using α from the cylinder's equation in the constraint, we find

$$a_y = -\alpha R = -\frac{2T}{MR}R = -\frac{2T}{M}$$

Thus the tension is $T = -\frac{1}{2}Ma_y$. If we use this value of the tension in the bucket's equation, we can solve for the acceleration:

$$ma_y = -\tfrac{1}{2}Ma_y - mg$$

$$a_y = -\frac{g}{(1 + M/2m)} = -7.84 \text{ m/s}^2$$

The time to fall through $\Delta y = -1.0$ m is found from kinematics:

$$\Delta y = \tfrac{1}{2}a_y(\Delta t)^2$$

$$\Delta t = \sqrt{\frac{2\,\Delta y}{a_y}} = \sqrt{\frac{2(-1.0\text{ m})}{-7.84\text{ m/s}^2}} = 0.50 \text{ s}$$

ASSESS The expression for the acceleration gives $a_y = -g$ if $M = 0$. This makes sense because the bucket would be in free fall if there were no cylinder. When the cylinder has mass, the downward force of gravity on the bucket has to accelerate the bucket *and* spin the cylinder. Consequently, the acceleration is reduced and the bucket takes longer to fall.

12.8 Static Equilibrium

An extended object that is completely stationary is in *static equilibrium*. It has no linear acceleration ($\vec{a} = \vec{0}$) and no angular acceleration ($\alpha = 0$). Thus, from Newton's laws, the conditions for static equilibrium are no net force *and* no net torque. These two rules are the basis for a branch of engineering, called *statics,* that analyzes buildings, dams, bridges, and other structures in static equilibrium.

Section 6.1 introduced the model of mechanical equilibrium for objects that can be represented as particles. For extended objects, we have the **static equilibrium model.**

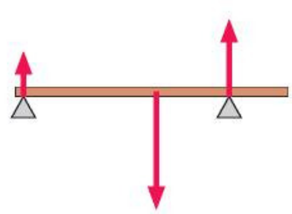
Structures such as bridges are analyzed in engineering statics.

MODEL 12.3

Static equilibrium

For extended objects at rest.

- Model the object as a rigid body with no acceleration.
- Mathematically:
  - No net force: $\vec{F}_{\text{net}} = \sum \vec{F}_i = \vec{0}$, and
  - No net torque: $\tau_{\text{net}} = \sum \tau_i = 0$
- The torque is zero about *every* point, so use any point that is convenient for the pivot point.
- Limitations: Model fails if either the forces or the torques aren't balanced.



$\vec{a} = \vec{0}$ $\alpha = 0$

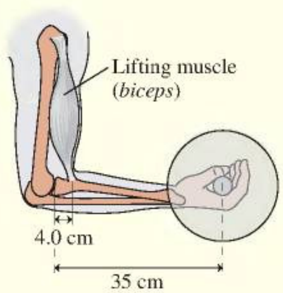
The object is at rest.

For any point you choose, an object that is not rotating is not rotating about that point. This seems to be a trivial statement, but it has an important implication: For a rigid body in static equilibrium, the net torque is zero about *any* point. You can use any point you wish as the pivot point for calculating torque. Even so, some choices are better than others for problem solving. As the examples will show, it's often best to choose a point at which several forces act because the torques exerted by those forces will be zero.

EXAMPLE 12.13 Lifting weights

Weightlifting can exert extremely large forces on the body's joints and tendons. In the *strict curl* event, a standing athlete uses both arms to lift a barbell by moving only his forearms, which pivot at the elbows. The record weight lifted in the strict curl is over 200 pounds (about 900 N). **FIGURE 12.31** shows the arm bones and the biceps, the main lifting muscle when the forearm is horizontal. What is the tension in the tendon connecting the biceps muscle to the bone while a 900 N barbell is held stationary in this position?

FIGURE 12.31 An arm holding a barbell.



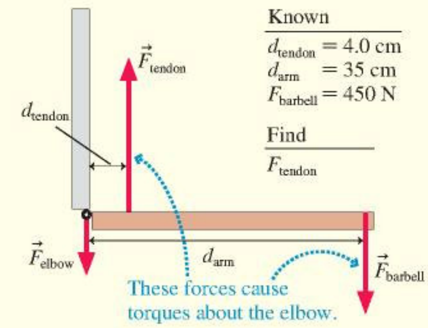
MODEL Model the arm as two rigid rods connected by a hinge. We'll ignore the arm's weight because it is so much less than that of the barbell. Although the tendon pulls at a slight angle, it is close enough to vertical that we'll treat it as such.

VISUALIZE **FIGURE 12.32** shows the forces acting on our simplified model of the forearm. The biceps pulls the forearm up against the upper arm at the elbow, so the force $\vec{F}_{elbow}$ *on* the forearm at the elbow—a force due to the upper arm—is a downward force.

SOLVE Static equilibrium requires both the net force *and* the net torque on the forearm to be zero. Only the y-component of force is relevant, and setting it to zero gives a first equation:

$$\sum F_y = F_{tendon} - F_{elbow} - F_{barbell} = 0$$

FIGURE 12.32 A pictorial representation of the forces involved.



Because each arm supports half the weight of the barbell, $F_{barbell} = 450$ N. We don't know either F_{tendon} or F_{elbow}, nor does the force equation give us enough information to find them. But the fact that the net torque also must be zero gives us that extra information. The torque is zero about *every* point, so we can choose any point we wish to calculate the torque. The elbow joint is a convenient point because force $\vec{F}_{elbow}$ exerts no torque about this point; its moment arm is zero. Thus the torque equation is

$$\tau_{net} = d_{tendon} F_{tendon} - d_{arm} F_{barbell} = 0$$

The tension in the tendon tries to rotate the arm ccw, so it produces a positive torque. Similarly, the torque due to the barbell is negative. We can solve the torque equation for F_{tendon} to find

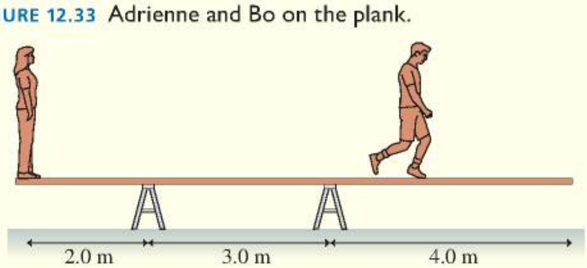
$$F_{tendon} = F_{barbell} \frac{d_{arm}}{d_{tendon}} = (450 \text{ N}) \frac{35 \text{ cm}}{4.0 \text{ cm}} = 3900 \text{ N}$$

ASSESS The short distance d_{tendon} from the tendon to the elbow joint means that the force supplied by the biceps has to be very large to counter the torque generated by a force applied at the opposite end of the forearm. Although we ended up not needing the force equation in this problem, we could now use it to calculate that the force exerted at the elbow is $F_{elbow} = 3450$ N. These large forces can easily damage the tendon or the elbow.

EXAMPLE 12.14 Walking the plank

Adrienne (50 kg) and Bo (90 kg) are playing on a 100 kg rigid plank resting on the supports seen in **FIGURE 12.33**. If Adrienne stands on the left end, can Bo walk all the way to the right end without the plank tipping over? If not, how far can he get past the support on the right?

FIGURE 12.33 Adrienne and Bo on the plank.



MODEL Model the plank as a uniform rigid body with its center of mass at the center.

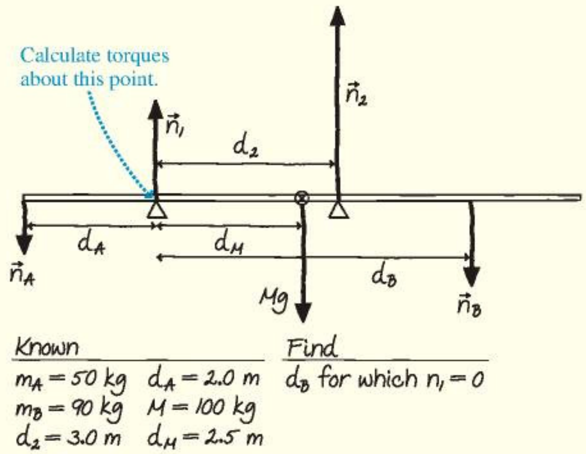
VISUALIZE **FIGURE 12.34** shows the forces acting on the plank. Both supports exert upward forces. $\vec{n}_A$ and $\vec{n}_B$ are the normal forces of Adrienne's and Bo's feet pushing down on the board.

SOLVE Because the plank is resting on the supports, not held down, forces $\vec{n}_1$ and $\vec{n}_2$ must point upward. (The supports could pull down if the plank were nailed to them, but that's not the case here.) Force $\vec{n}_1$ will decrease as Bo moves to the right, and the tipping point occurs when $n_1 = 0$. The plank remains in static equilibrium right up to the tipping point, so both the net force and the net torque on it are zero. The force equation is

$$\sum F_y = n_1 + n_2 - n_A - n_B - Mg$$
$$= n_1 + n_2 - m_A g - m_B g - Mg = 0$$

Adrienne is at rest, with zero net force, so her downward force on the board, an action/reaction pair with the upward normal force of the board on her, equals her weight: $n_A = m_A g$. Bo's center of

FIGURE 12.34 A pictorial representation of the forces on the plank.



mass oscillates up and down as he walks, so he's *not* in equilibrium and, strictly speaking, $n_B \neq m_B g$. But we'll assume that he edges out onto the board slowly, with minimal bouncing, in which case $n_B = m_B g$ is a reasonable approximation.

We can again choose any point we wish for calculating torque. Let's use the support on the left. Adrienne and the support on the right exert positive torques about this point; the other forces exert negative torques. Force $\vec{n}_1$ exerts no torque because it acts at the pivot point. Thus the torque equation is

$$\tau_{net} = d_A m_A g - d_B m_B g - d_M M g + d_2 n_2 = 0$$

At the tipping point, where $n_1 = 0$, the force equation gives $n_2 = (m_A + m_B + M)g$. Substituting this into the torque equation and then solving for Bo's position give

$$d_B = \frac{d_A m_A - d_M M + d_2(m_A + m_B + M)}{m_B} = 6.3 \text{ m}$$

Bo doesn't quite make it to the end. The plank tips when he's 6.3 m past the left support, our pivot point, and thus 3.3 m past the support on the right.

ASSESS We could have solved this problem somewhat more simply had we chosen the support on the right for calculating the torques. However, you might not recognize the "best" point for calculating the torques in a problem. The point of this example is that it doesn't matter which point you choose.

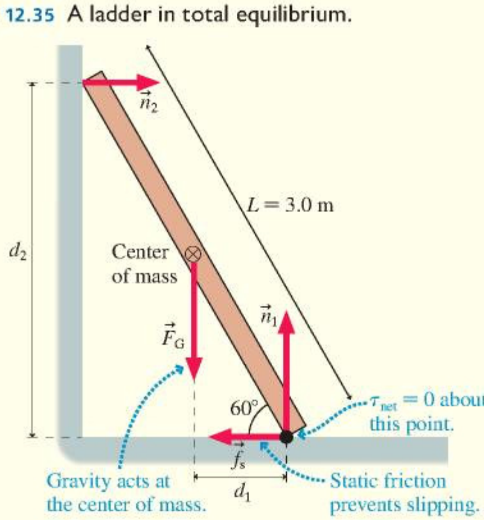
EXAMPLE 12.15 Will the ladder slip?

A 3.0-m-long ladder leans against a frictionless wall at an angle of 60°. What is the minimum value of μ_s, the coefficient of static friction with the ground, that prevents the ladder from slipping?

MODEL The ladder is a rigid rod of length L. To not slip, it must be in both translational equilibrium ($\vec{F}_{net} = \vec{0}$) and rotational equilibrium ($\tau_{net} = 0$).

VISUALIZE FIGURE 12.35 shows the ladder and the forces acting on it.

FIGURE 12.35 A ladder in total equilibrium.



SOLVE The x- and y-components of $\vec{F}_{net} = \vec{0}$ are

$$\sum F_x = n_2 - f_s = 0$$
$$\sum F_y = n_1 - Mg = 0$$

The net torque is zero about *any* point, so which should we choose? The bottom corner of the ladder is a good choice because two forces pass through this point and have no torque about it. The torque about the bottom corner is

$$\tau_{net} = d_1 F_G - d_2 n_2 = \tfrac{1}{2}(L\cos 60°)Mg - (L\sin 60°)n_2 = 0$$

The signs are based on the observation that $\vec{F}_G$ would cause the ladder to rotate ccw while $\vec{n}_2$ would cause it to rotate cw. All together, we have three equations in the three unknowns n_1, n_2, and f_s. If we solve the third for n_2,

$$n_2 = \frac{\frac{1}{2}(L\cos 60°)Mg}{L\sin 60°} = \frac{Mg}{2\tan 60°}$$

we can then substitute this into the first to find

$$f_s = \frac{Mg}{2\tan 60°}$$

Our model of friction is $f_s \le f_{s\,max} = \mu_s n_1$. We can find n_1 from the second equation: $n_1 = Mg$. Using this, the model of static friction tells us that

$$f_s \le \mu_s Mg$$

Comparing these two expressions for f_s, we see that μ_s must obey

$$\mu_s \ge \frac{1}{2\tan 60°} = 0.29$$

Thus the minimum value of the coefficient of static friction is 0.29.

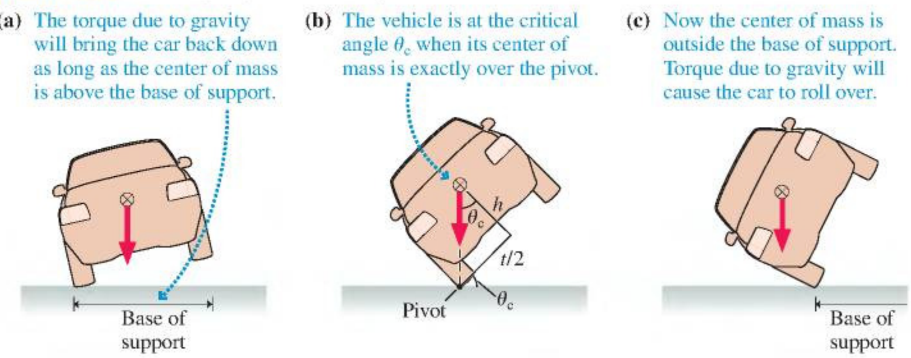
ASSESS You know from experience that you can lean a ladder or other object against a wall if the ground is "rough," but it slips if the surface is too smooth. 0.29 is a "medium" value for the coefficient of static friction, which is reasonable.

Balance and Stability

If you tilt a box up on one edge by a small amount and let go, it falls back down. If you tilt it too much, it falls over. And if you tilt "just right," you can get the box to balance on its edge. What determines these three possible outcomes?

FIGURE 12.36 illustrates the idea with a car, but the results are general and apply in many situations. As long as the object's center of mass remains over the base of support, torque due to gravity will rotate the object back to its equilibrium position.

FIGURE 12.36 Stability depends on the position of the center of mass.



This dancer balances *en pointe* by having her center of mass directly over her toes, her base of support.

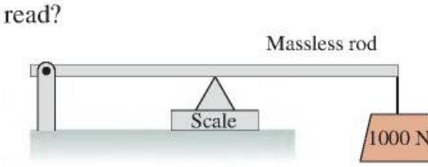
A *critical angle* θ_c is reached when the center of mass is directly over the pivot point. This is the point of balance, with no net torque. For vehicles, the distance between the tires is called the track width t. If the height of the center of mass is h, you can see from Figure 12.36b that the critical angle is

$$\theta_c = \tan^{-1}\left(\frac{t}{2h}\right)$$

For passenger cars with $h \approx 0.33t$, the critical angle is $\theta_c \approx 57°$. But for a sport utility vehicle (SUV) with $h \approx 0.47t$, a higher center of mass, the critical angle is only $\theta_c \approx 47°$. Various automobile safety groups have determined that a vehicle with $\theta_c > 50°$ is unlikely to roll over in an accident. A rollover becomes increasingly likely when θ_c is reduced below 50°. The general rule is that **a wider base of support and/or a lower center of mass improve stability.**

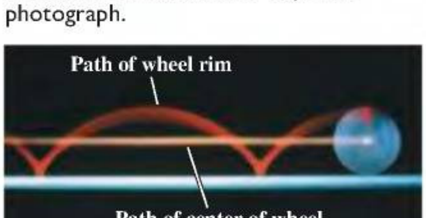
STOP TO THINK 12.6 What does the scale read?

a. 500 N
b. 1000 N
c. 2000 N
d. 4000 N



12.9 Rolling Motion

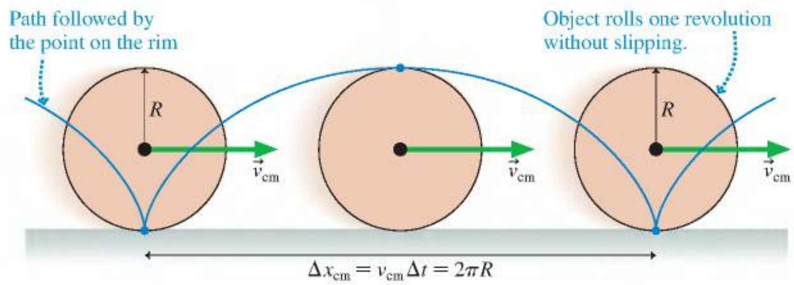
FIGURE 12.37 The trajectories of the center of a wheel and of a point on the rim are seen in a time-exposure photograph.



Rolling is a *combination motion* in which an object rotates about an axis that is moving along a straight-line trajectory. For example, FIGURE 12.37 is a time-exposure photo of a rolling wheel with one lightbulb on the axis and a second lightbulb at the edge. The axis light moves straight ahead, but the edge light moves along a curve. Let's see if we can understand this interesting motion. We'll consider only objects that roll without slipping.

FIGURE 12.38 shows a round object—a wheel or a sphere—that rolls forward exactly one revolution. The point that had been on the bottom follows the curve you saw in Figure 12.37 to the top and back to the bottom. *Because the object doesn't slip,* the center of mass moves forward exactly one circumference: $\Delta x_{cm} = 2\pi R$.

We can also write the distance traveled in terms of the velocity of the center of mass: $\Delta x_{cm} = v_{cm}\,\Delta t$. But Δt, the time it takes the object to make one complete revolution, is nothing other than the rotation period T. In other words, $\Delta x_{cm} = v_{cm}T$.



◀ **FIGURE 12.38** An object rolling through one revolution.

These two expressions for Δx_{cm} come from two perspectives on the motion: one looking at the rotation and the other looking at the translation of the center of mass. But it's the same distance no matter how you look at it, so these two expressions must be equal. Consequently,

$$\Delta x_{cm} = 2\pi R = v_{cm} T \tag{12.34}$$

If we divide by T, we can write the center-of-mass velocity as

$$v_{cm} = \frac{2\pi}{T} R \tag{12.35}$$

But $2\pi/T$ is the angular velocity ω, as you learned in Chapter 4, leading to

$$v_{cm} = R\omega \tag{12.36}$$

Equation 12.36 is the **rolling constraint,** the basic link between translation and rotation for objects that roll without slipping.

Let's look carefully at a particle in the rolling object. As **FIGURE 12.39a** shows, the position vector $\vec{r}_i$ for particle i is the vector sum $\vec{r}_i = \vec{r}_{cm} + \vec{r}_{i,\,rel}$. Taking the time derivative of this equation, we can write the velocity of particle i as

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_{i,\,rel} \tag{12.37}$$

In other words, the velocity of particle i can be divided into two parts: the velocity $\vec{v}_{cm}$ of the object as a whole plus the velocity $\vec{v}_{i,\,rel}$ of particle i relative to the center of mass (i.e., the velocity that particle i would have if the object were only rotating and had no translational motion).

FIGURE 12.39b applies this idea to point P at the very bottom of the rolling object, the point of contact between the object and the surface. This point is moving around the center of the object at angular velocity ω, so $v_{i,\,rel} = -R\omega$. The negative sign indicates that the motion is cw. At the same time, the center-of-mass velocity, Equation 12.36, is $v_{cm} = R\omega$. Adding these, we find that the velocity of point P, the lowest point, is $v_i = 0$. In other words, **the point on the bottom of a rolling object is instantaneously at rest.**

Although this seems surprising, it is really what we mean by "rolling without slipping." If the bottom point had a velocity, it would be moving horizontally relative to the surface. In other words, it would be slipping or sliding across the surface. To roll without slipping, the bottom point, the point touching the surface, must be at rest.

FIGURE 12.40 shows how the velocity vectors at the top, center, and bottom of a rotating wheel are found by adding the rotational velocity vectors to the center-of-mass velocity. You can see that $v_{bottom} = 0$ and that $v_{top} = 2R\omega = 2v_{cm}$.

FIGURE 12.39 The motion of a particle in the rolling object.

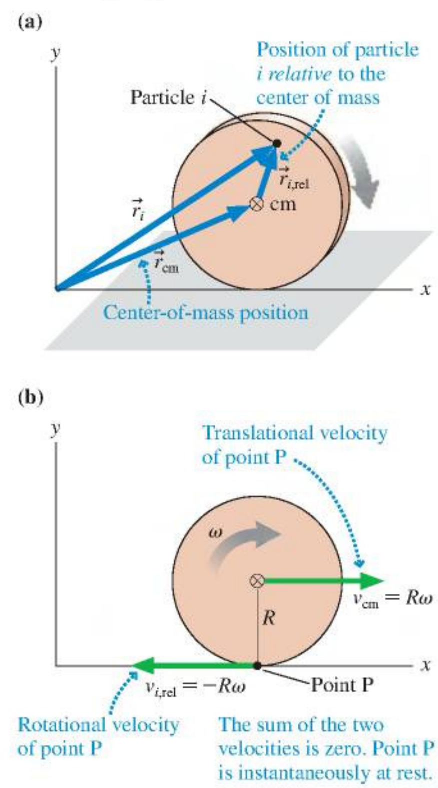
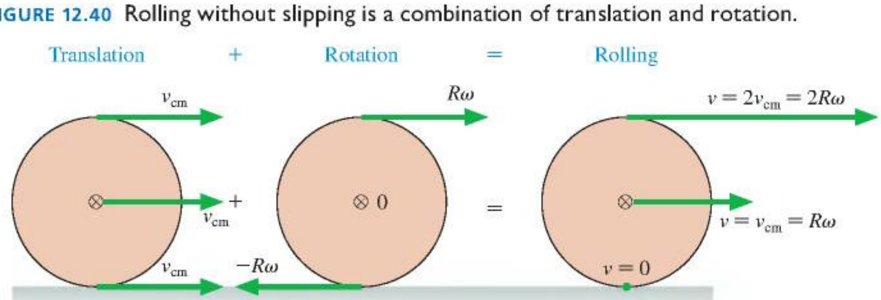


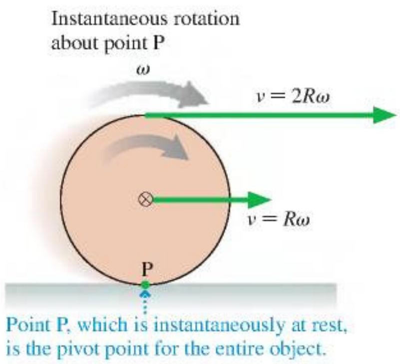
FIGURE 12.40 Rolling without slipping is a combination of translation and rotation.



Kinetic Energy of a Rolling Object

We found earlier that the rotational kinetic energy of a rigid body in pure rotational motion is $K_{rot} = \frac{1}{2}I\omega^2$. Now we would like to find the kinetic energy of an object that rolls without slipping, a combination of rotational and translation motion.

FIGURE 12.41 Rolling motion is an instantaneous rotation about point P.



We begin with the observation that the bottom point in FIGURE 12.41 is instantaneously at rest. Consequently, we can think of an axis through P as an *instantaneous axis of rotation*. The idea of an instantaneous axis of rotation seems a little far-fetched, but it is confirmed by looking at the instantaneous velocities of the center point and the top point. We found these in Figure 12.40 and they are shown again in Figure 12.41. They are exactly what you would expect as the tangential velocity $v_t = r\omega$ for rotation about P at distances R and $2R$.

From this perspective, the object's motion is pure rotation about point P. Thus the kinetic energy is that of pure rotation:

$$K = K_{\text{rotation about P}} = \tfrac{1}{2}I_P\omega^2 \tag{12.38}$$

I_P is the moment of inertia for rotation about point P. We can use the parallel-axis theorem to write I_P in terms of the moment of inertia I_{cm} about the center of mass. Point P is displaced by distance $d = R$; thus

$$I_P = I_{cm} + MR^2$$

Using this expression in Equation 12.38 gives us the kinetic energy:

$$K = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}M(R\omega)^2 \tag{12.39}$$

We know from the rolling constraint that $R\omega$ is the center-of-mass velocity v_{cm}. Thus the kinetic energy of a rolling object is

$$K_{rolling} = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv_{cm}^2 = K_{rot} + K_{cm} \tag{12.40}$$

In other words, **the rolling motion of a rigid body can be described as a translation of the center of mass (with kinetic energy K_{cm}) plus a rotation about the center of mass (with kinetic energy K_{rot}).**

The Great Downhill Race

FIGURE 12.42 Which will win the downhill race?

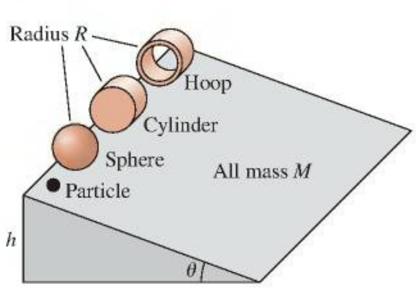


FIGURE 12.42 shows a contest in which a sphere, a cylinder, and a circular hoop, all of mass M and radius R, are placed at height h on a slope of angle θ. All three are released from rest at the same instant of time and roll down the ramp without slipping. To make things more interesting, they are joined by a particle of mass M that slides down the ramp without friction. Which one will win the race to the bottom of the hill? Does rotation affect the outcome?

An object's initial gravitational potential energy is transformed into kinetic energy as it rolls (or slides, in the case of the particle). The kinetic energy, as we just discovered, is a combination of translational and rotational kinetic energy. If we choose the bottom of the ramp as the zero point of potential energy, the statement of energy conservation $K_f = U_i$ can be written

$$\tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv_{cm}^2 = Mgh \tag{12.41}$$

The translational and rotational velocities are related by $\omega = v_{cm}/R$. In addition, notice from Table 12.2 that the moments of inertia of all the objects can be written in the form

$$I_{cm} = cMR^2 \tag{12.42}$$

where c is a constant that depends on the object's geometry. For example, $c = \frac{2}{5}$ for a sphere but $c = 1$ for a circular hoop. Even the particle can be represented by $c = 0$, which eliminates the rotational kinetic energy.

With this information, Equation 12.41 becomes

$$\tfrac{1}{2}(cMR^2)\left(\frac{v_{cm}}{R}\right)^2 + \tfrac{1}{2}Mv_{cm}^2 = \tfrac{1}{2}M(1+c)v_{cm}^2 = Mgh$$

Thus the finishing speed of an object with $I = cMR^2$ is

$$v_{cm} = \sqrt{\frac{2gh}{1 + c}} \tag{12.43}$$

The final speed is independent of both M and R, but it does depend on the *shape* of the rolling object. The particle, with the smallest value of c, will finish with the highest speed, while the circular hoop, with the largest c, will be the slowest. In other words, the rolling aspect of the motion *does* matter!

We can use Equation 12.43 to find the acceleration a_{cm} of the center of mass. The objects move through distance $\Delta x = h/\sin\theta$, so we can use constant-acceleration kinematics to find

$$\begin{aligned} {v_{cm}}^2 &= 2a_{cm}\,\Delta x \\ a_{cm} &= \frac{{v_{cm}}^2}{2\Delta x} = \frac{2gh/(1+c)}{2h/\sin\theta} = \frac{g\sin\theta}{1+c} \end{aligned} \tag{12.44}$$

Recall, from Chapter 2, that $a_{particle} = g\sin\theta$ is the acceleration of a particle sliding down a frictionless incline. We can use this fact to write Equation 12.44 in an interesting form:

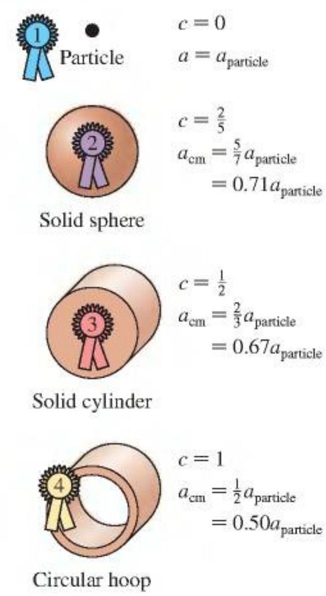
$$a_{cm} = \frac{a_{particle}}{1 + c} \tag{12.45}$$

This analysis leads us to the conclusion that **the acceleration of a rolling object is less—in some cases significantly less—than the acceleration of a particle.** The reason is that the energy has to be shared between translational kinetic energy and rotational kinetic energy. A particle, by contrast, can put all its energy into translational kinetic energy.

FIGURE 12.43 shows the results of the race. The simple particle wins by a fairly wide margin. Of the solid objects, the sphere has the largest acceleration. Even so, its acceleration is only 71% the acceleration of a particle. The acceleration of the circular hoop, which comes in last, is a mere 50% that of a particle.

NOTE The objects having the largest acceleration are those whose mass is most concentrated near the center. Placing the mass far from the center, as in the hoop, increases the moment of inertia. Thus it requires a larger effort to get a hoop rolling than to get a sphere of equal mass rolling.

FIGURE 12.43 And the winner is ...



12.10 The Vector Description of Rotational Motion

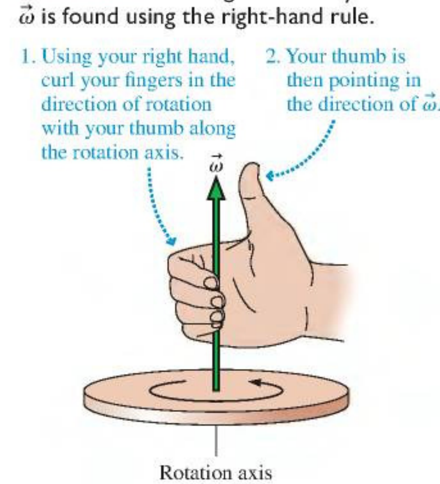
Rotation about a fixed axis, such as an axle, can be described in terms of a scalar angular velocity ω and a scalar torque τ, using a plus or minus sign to indicate the direction of rotation. This is very much analogous to the one-dimensional kinematics of Chapter 2. For more general rotational motion, angular velocity, torque, and other quantities must be treated as *vectors*. We won't go into much detail because the subject rapidly gets very complicated, but we will sketch some important basic ideas.

The Angular Velocity Vector

FIGURE 12.44 shows a rotating rigid body. We can define an angular velocity vector $\vec{\omega}$ as follows:

- The magnitude of $\vec{\omega}$ is the object's angular velocity ω.
- $\vec{\omega}$ points along the axis of rotation in the direction given by the *right-hand rule* illustrated in Figure 12.44.

FIGURE 12.44 The angular velocity vector $\vec{\omega}$ is found using the right-hand rule.



If the object rotates in the xy-plane, the vector $\vec{\omega}$ points along the z-axis. The scalar angular velocity $\omega = v_t/r$ that we've been using is now seen to be ω_z, the z-component of the vector $\vec{\omega}$. You should convince yourself that the sign convention for ω (positive for ccw rotation, negative for cw rotation) is equivalent to having the vector $\vec{\omega}$ pointing in the positive z-direction or the negative z-direction.

The Cross Product of Two Vectors

We defined the torque exerted by force $\vec{F}$ to be $\tau = rF\sin\phi$. The quantity F is the magnitude of the force vector $\vec{F}$, and the distance r is really the magnitude of the position vector $\vec{r}$. Hence torque looks very much like a product of the two vectors $\vec{r}$ and $\vec{F}$. Previously, in conjunction with the definition of work, we introduced the dot product of two vectors: $\vec{A}\cdot\vec{B} = AB\cos\alpha$, where α is the angle between the vectors. $\tau = rF\sin\phi$ is a different way of multiplying vectors that depends on the *sine* of the angle between them.

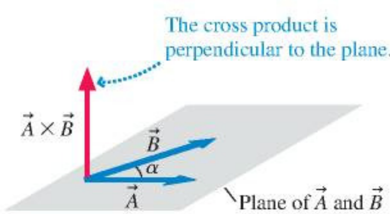


FIGURE 12.45 The cross product $\vec{A}\times\vec{B}$, is a vector perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$.

FIGURE 12.45 shows two vectors, $\vec{A}$ and $\vec{B}$, with angle α between them. We define the **cross product** of $\vec{A}$ and $\vec{B}$ as the vector

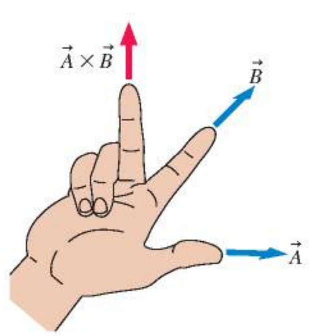
$$\vec{A}\times\vec{B} \equiv (AB\sin\alpha, \text{ in the direction given by the right-hand rule}) \qquad (12.46)$$

The symbol $\times$ between the vectors is *required* to indicate a cross product. The cross product is also called the **vector product** because the result is a vector.

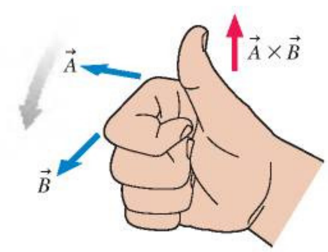
The **right-hand rule,** which specifies the direction of $\vec{A}\times\vec{B}$, can be stated in three different but equivalent ways:

Using the right-hand rule

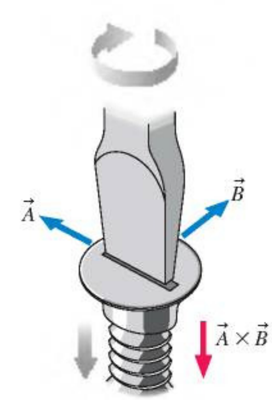
Spread your *right* thumb and index finger apart by angle α. Bend your middle finger so that it is *perpendicular* to your thumb and index finger. Orient your hand so that your thumb points in the direction of $\vec{A}$ and your index finger in the direction of $\vec{B}$. Your middle finger now points in the direction of $\vec{A}\times\vec{B}$.



Make a loose fist with your *right* hand with your thumb extended outward. Orient your hand so that your thumb is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and your fingers are curling *from* the line of vector $\vec{A}$ *toward* the line of vector $\vec{B}$. Your thumb now points in the direction of $\vec{A}\times\vec{B}$.



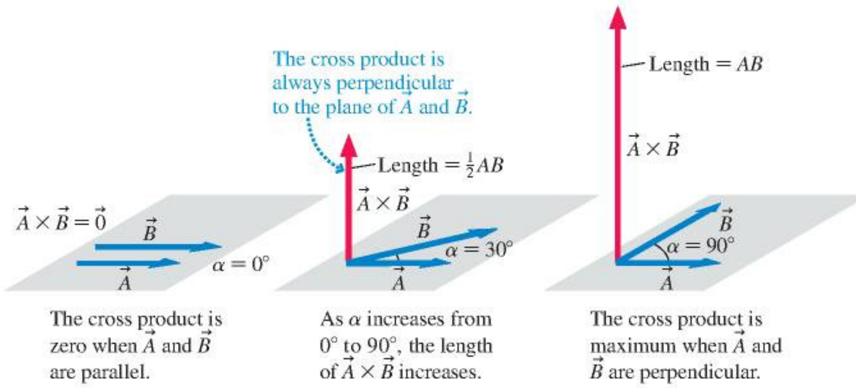
Imagine using a screwdriver to turn the slot in the head of a screw from the direction of $\vec{A}$ to the direction of $\vec{B}$. The screw will move either "in" or "out." The direction in which the screw moves is the direction of $\vec{A}\times\vec{B}$.



These methods are easier to demonstrate than to describe in words! Your instructor will show you how they work. Some individuals find one method of thinking about the direction of the cross product easier than the others, but they all work, and you'll soon find the method that works best for you.

Referring back to Figure 12.45, you should use the right-hand rule to convince yourself that the cross product $\vec{A}\times\vec{B}$ is a vector that points *upward,* perpendicular to the plane of $\vec{A}$ and $\vec{B}$. **FIGURE 12.46** shows that the cross product, like the dot product, depends on the angle between the two vectors. Notice the two special cases: $\vec{A}\times\vec{B} = \vec{0}$ when $\alpha = 0°$ (parallel vectors) and $\vec{A}\times\vec{B}$ has its maximum magnitude AB when $\alpha = 90°$ (perpendicular vectors).

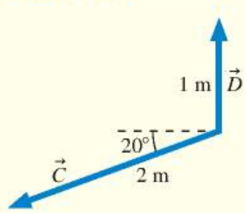
FIGURE 12.46 The magnitude of the cross-product vector increases from 0 to AB as α increases from 0° to 90°.



EXAMPLE 12.16 Calculating a cross product

FIGURE 12.47 shows vectors $\vec{C}$ and $\vec{D}$ in the plane of the page. What is the cross product $\vec{E} = \vec{C}\times\vec{D}$?

FIGURE 12.47 Vectors $\vec{C}$ and $\vec{D}$.



SOLVE The angle between the two vectors is $\alpha = 110°$. Consequently, the magnitude of the cross product is

$$E = CD\sin\alpha = (2\text{ m})(1\text{ m})\sin(110°) = 1.88\text{ m}^2$$

The direction of $\vec{E}$ is given by the right-hand rule. To curl your right fingers from $\vec{C}$ to $\vec{D}$, you have to point your thumb *into* the page. Alternatively, if you turned a screwdriver from $\vec{C}$ to $\vec{D}$ you would be driving a screw *into* the page. Thus

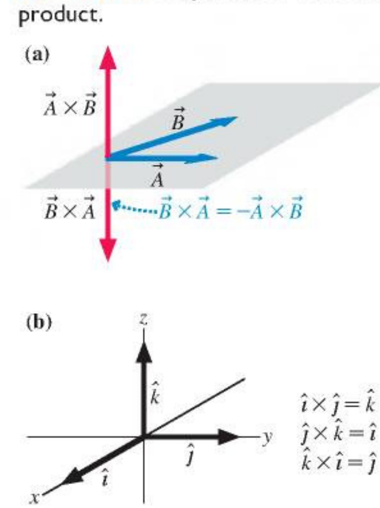
$$\vec{E} = (1.88\text{ m}^2,\text{ into page})$$

ASSESS Notice that $\vec{E}$ has units of m^2.

The cross product has three important properties:

1. The product $\vec{A}\times\vec{B}$ is *not* equal to the product $\vec{B}\times\vec{A}$. That is, the cross product does not obey the commutative rule $ab = ba$ that you know from arithmetic. In fact, you can see from the right-hand rule that the product $\vec{B}\times\vec{A}$ points in exactly the opposite direction from $\vec{A}\times\vec{B}$. Thus, as FIGURE 12.48a shows,
$$\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$$
2. In a *right-handed coordinate system,* which is the standard coordinate system of science and engineering, the z-axis is oriented relative to the xy-plane such that the unit vectors obey $\hat{\imath}\times\hat{\jmath} = \hat{k}$. This is shown in FIGURE 12.48b. You can also see from this figure that $\hat{\jmath}\times\hat{k} = \hat{\imath}$ and $\hat{k}\times\hat{\imath} = \hat{\jmath}$.
3. The derivative of a cross product is
$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt} \qquad (12.47)$$

FIGURE 12.48 Properties of the cross product.



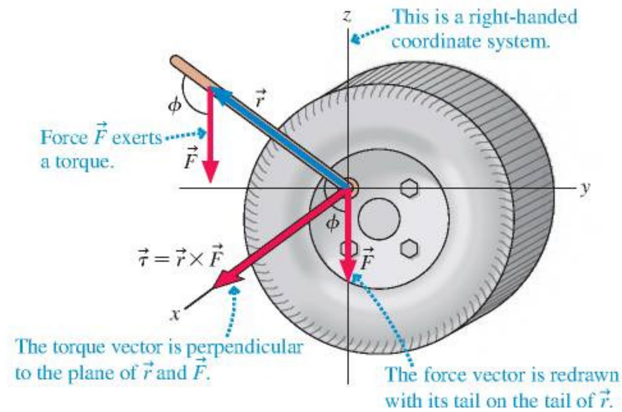
Torque

Now let's return to torque. As a concrete example, FIGURE 12.49 on the next page shows a long wrench being used to loosen the nuts holding a car wheel on. We've established a right-handed coordinate system with its origin at the nut, so force $\vec{F}$ exerts a torque about the origin. Let's define a *torque vector*

$$\vec{\tau} \equiv \vec{r}\times\vec{F} \qquad (12.48)$$

If we place the vector tails together in order to use the right-hand rule, we see that the torque vector is perpendicular to the plane of $\vec{r}$ and $\vec{F}$. The angle between the vectors is ϕ, so the magnitude of the torque is $\tau = rF|\sin\phi|$.

FIGURE 12.49 The torque vector.



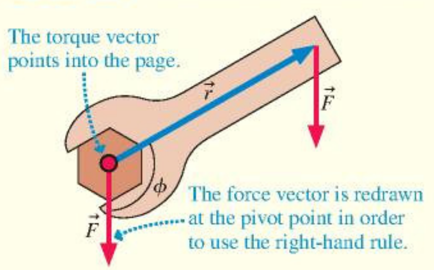
You can see that the scalar torque $\tau = rF\sin\phi$ we've been using is really the component along the rotation axis—in this case τ_x—of the vector $\vec{\tau}$. This is the basis for our earlier sign convention for τ. In Figure 12.49, where the force causes a ccw rotation, the torque vector points in the positive x-direction, and thus τ_x is positive.

EXAMPLE 12.17 Wrench torque revisited

Example 12.8 found the torque that Luis exerts on a nut by pulling on the end of a wrench. What is the torque vector?

VISUALIZE FIGURE 12.50 shows the position vector $\vec{r}$, drawn from the pivot point to the point where the force is applied. The figure also redraws the force vector $\vec{F}$ at the pivot point, not because force is applied there but because it's easiest to use the right-hand rule if the vectors are drawn with their tails together.

FIGURE 12.50 Calculating the torque vector.



SOLVE We already know the magnitude of the torque, 17 N m, from Example 12.8. Now we need to apply the right-hand rule. If you place your right thumb along $\vec{r}$ and your index finger along $\vec{F}$, which is somewhat awkward, you'll see that your middle finger points into the page. Alternatively, make a loose fist of your right hand, then orient your fist so that your fingers curl *from* $\vec{r}$ *toward* $\vec{F}$. Doing so requires your thumb to point into the page. Using either method, we conclude that

$$\vec{\tau} = (17\ \text{N m, into page})$$

12.11 Angular Momentum

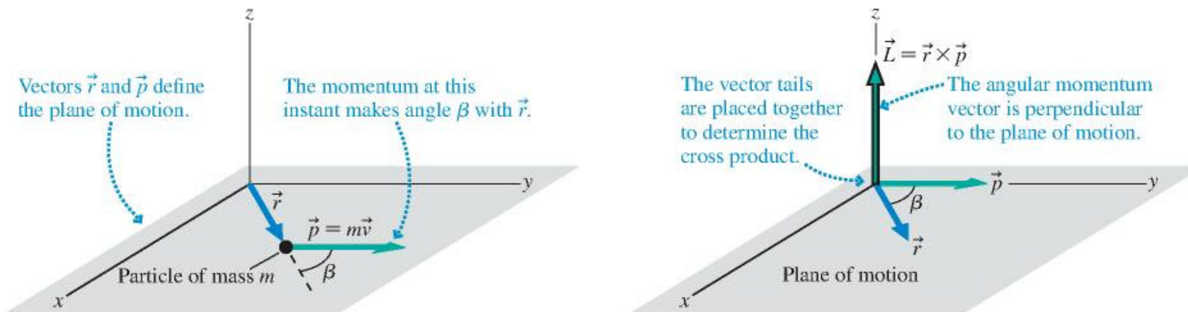
FIGURE 12.51 shows a particle that, at this instant, is located at position $\vec{r}$ and is moving with momentum $\vec{p} = m\vec{v}$. Together, $\vec{r}$ and $\vec{p}$ define the *plane of motion*. We define the particle's **angular momentum** $\vec{L}$ relative to the origin to be the vector

$$\vec{L} \equiv \vec{r} \times \vec{p} = (mrv\sin\beta,\ \text{direction of right-hand rule}) \qquad (12.49)$$

Because of the cross product, **the angular momentum vector is perpendicular to the plane of motion.** The units of angular momentum are kg m^2/s.

NOTE Angular momentum is the rotational equivalent of linear momentum in much the same way that torque is the rotational equivalent of force. Notice that the vector definitions are parallel: $\vec{\tau} \equiv \vec{r} \times \vec{F}$ and $\vec{L} \equiv \vec{r} \times \vec{p}$.

FIGURE 12.51 The angular momentum vector $\vec{L}$.

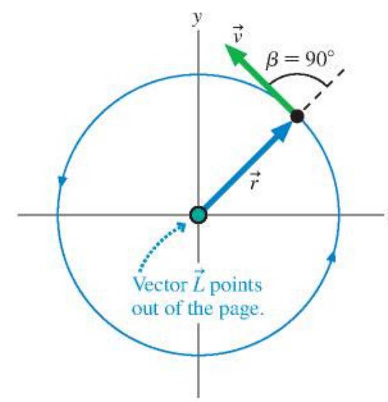


Angular momentum, like torque, is *about* the point from which $\vec{r}$ is measured. A different origin would yield a different angular momentum. Angular momentum is especially simple for a particle in circular motion. As FIGURE 12.52 shows, the angle β between $\vec{p}$ (or $\vec{v}$) and $\vec{r}$ is always 90° if we make the obvious choice of measuring $\vec{r}$ from the center of the circle. For motion in the xy-plane, the angular momentum vector $\vec{L}$—which must be perpendicular to the plane of motion—is entirely along the z-axis:

$$L_z = mrv_t \qquad \text{(particle in circular motion)} \qquad (12.50)$$

where v_t is the tangential component of velocity. Our sign convention for v_t makes L_z, like ω, positive for a ccw rotation, negative for a cw rotation.

FIGURE 12.52 Angular momentum of circular motion.



In Chapter 11, we found that Newton's second law for a particle can be written $\vec{F}_{net} = d\vec{p}/dt$. There's a similar connection between torque and angular momentum. To show this, we take the time derivative of $\vec{L}$:

$$\begin{aligned}\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) &= \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} \\ &= \vec{v}\times\vec{p} + \vec{r}\times\vec{F}_{net}\end{aligned} \qquad (12.51)$$

where we used Equation 12.47 for the derivative of a cross product. We also used the definitions $\vec{v} = d\vec{r}/dt$ and $\vec{F}_{net} = d\vec{p}/dt$.

Vectors $\vec{v}$ and $\vec{p}$ are parallel, and the cross product of two parallel vectors is $\vec{0}$. Thus the first term in Equation 12.51 vanishes. The second term $\vec{r}\times\vec{F}_{net}$ is the net torque, $\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 + \cdots$, so we arrive at

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{net} \qquad (12.52)$$

Equation 12.52, which says **a net torque causes the particle's angular momentum to change,** is the rotational equivalent of $d\vec{p}/dt = \vec{F}_{net}$.

Angular Momentum of a Rigid Body

Equation 12.52 is the angular momentum of a single particle. The angular momentum of a rigid body composed of particles with individual angular momenta $\vec{L}_1, \vec{L}_2, \vec{L}_3, \ldots$ is the vector sum

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots = \sum_i \vec{L}_i \qquad (12.53)$$

We can combine Equations 12.52 and 12.53 to find the rate of change of the system's angular momentum:

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{\tau}_i = \vec{\tau}_{net} \qquad (12.54)$$

Because any internal forces are action/reaction pairs of forces, acting with the same strength in opposite directions, the net torque due to internal forces is zero. Thus the only forces that contribute to the net torque are external forces exerted on the system by the environment.

For a system of particles, **the rate of change of the system's angular momentum is the net torque on the system.** Equation 12.54 is analogous to the Chapter 11 result $d\vec{P}/dt = \vec{F}_{net}$, which says that the rate of change of a system's total linear momentum is the net force on the system.

Conservation of Angular Momentum

A net torque on a rigid body causes its angular momentum to change. Conversely, the angular momentum does *not* change—it is *conserved*—for a system with no net torque. This is the basis of the law of conservation of angular momentum.

Law of conservation of angular momentum The angular momentum $\vec{L}$ of an isolated system ($\vec{\tau}_{net} = \vec{0}$) is conserved. The final angular momentum $\vec{L}_f$ is equal to the initial angular momentum $\vec{L}_i$. Both the magnitude *and* the direction of $\vec{L}$ are unchanged.

EXAMPLE 12.18 An expanding rod

Two equal masses are at the ends of a massless 50-cm-long rod. The rod spins at 2.0 rev/s about an axis through its midpoint. Suddenly, a compressed gas expands the rod out to a length of 160 cm. What is the rotation frequency after the expansion?

MODEL The forces push outward from the pivot and exert no torques. Thus the system's angular momentum is conserved.

VISUALIZE FIGURE 12.53 is a before-and-after pictorial representation. The angular momentum vectors $\vec{L}_i$ and $\vec{L}_f$ are perpendicular to the plane of motion.

SOLVE The particles are moving in circles, so each has angular momentum $L = mrv_t = mr^2\omega = \frac{1}{4}ml^2\omega$, where we used $r = \frac{1}{2}l$. Thus the initial angular momentum of the system is

$$L_i = \tfrac{1}{4}ml_i^2\omega_i + \tfrac{1}{4}ml_i^2\omega_i = \tfrac{1}{2}ml_i^2\omega_i$$

Similarly, the angular momentum after the expansion is $L_f = \frac{1}{2}ml_f^2\omega_f$. Angular momentum is conserved as the rod expands, thus

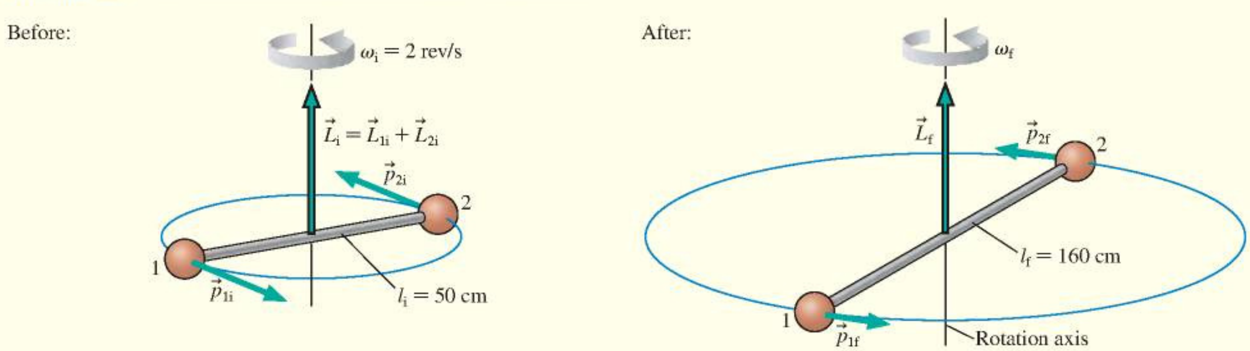
$$\tfrac{1}{2}ml_f^2\omega_f = \tfrac{1}{2}ml_i^2\omega_i$$

Solving for ω_f, we find

$$\omega_f = \left(\frac{l_i}{l_f}\right)^2 \omega_i = \left(\frac{50\text{ cm}}{160\text{ cm}}\right)^2 (2.0\text{ rev/s}) = 0.20\text{ rev/s}$$

ASSESS The values of the masses weren't needed. All that matters is the ratio of the lengths.

FIGURE 12.53 The system before and after the rod expands.



Angular Momentum and Angular Velocity

The analogy between linear and rotational motion has been so consistent that you might expect one more. The Chapter 9 result $\vec{P} = M\vec{v}$, which we can now write as $M\vec{v}_{cm}$ because it is translational motion of the object as a whole, might give us reason to anticipate that angular momentum and angular velocity are related by $\vec{L} = I\vec{\omega}$. Unfortunately, the analogy breaks down here. For an arbitrarily shaped object, the angular momentum vector and the angular velocity vector don't necessarily point in the same direction. The general relationship between $\vec{L}$ and $\vec{\omega}$ is beyond the scope of this text.

The good news is that the analogy *does* continue to hold in two important situations: the rotation of a *symmetrical* object about the symmetry axis and the rotation of any object about a fixed axle. For example, the axis of a cylinder or disk is a symmetry axis, as is any diameter through a sphere. In these two situations, the angular momentum and angular velocity are related by

$$\vec{L} = I\vec{\omega} \qquad \text{(rotation about a fixed axle or axis of symmetry)} \qquad (12.55)$$

This relationship is shown for a spinning disk in **FIGURE 12.54**. Equation 12.55 is particularly important for applying the law of conservation of angular momentum.

If an object's angular momentum is conserved, its angular speed is inversely proportional to its moment of inertia. The rotation of the rod in Example 12.18 slowed dramatically as it expanded because its moment of inertia increased. Similarly, the ice skater in **FIGURE 12.55** uses her moment of inertia to control her spin. She spins faster if she pulls in her arms, decreasing her moment of inertia. Similarly, extending her arms increases her moment of inertia, and her angular velocity drops until she can skate out of the spin. It's all a matter of conserving angular momentum.

TABLE 12.4 summarizes the analogies between linear and angular quantities.

TABLE 12.4 Angular and linear momentum and energy

Angular motion	Linear motion
$K_{rot} = \frac{1}{2}I\omega^2$	$K_{cm} = \frac{1}{2}Mv_{cm}{}^2$
$\vec{L} = I\vec{\omega}$ *	$\vec{P} = M\vec{v}_{cm}$
$d\vec{L}/dt = \vec{\tau}_{net}$	$d\vec{P}/dt = \vec{F}_{net}$
The angular momentum of a system is conserved if there is no net torque.	The linear momentum of a system is conserved if there is no net force.

*Rotation about an axis of symmetry.

FIGURE 12.54 The angular momentum vector about an axis of symmetry.

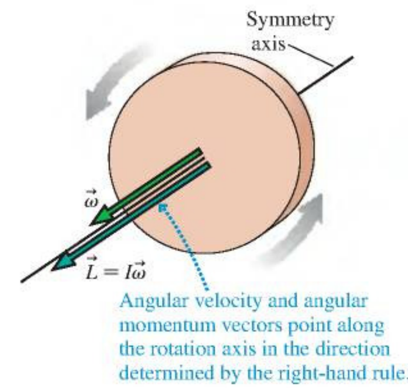
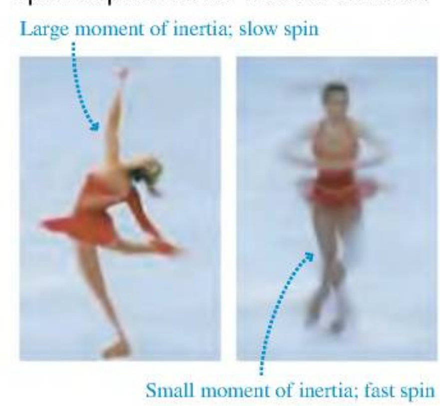


FIGURE 12.55 An ice skater's rotation speed depends on her moment of inertia.



EXAMPLE 12.19 Two interacting disks

A 20-cm-diameter, 2.0 kg solid disk is rotating at 200 rpm. A 20-cm-diameter, 1.0 kg circular loop is dropped straight down onto the rotating disk. Friction causes the loop to accelerate until it is "riding" on the disk. What is the final angular velocity of the combined system?

MODEL The friction between the two objects creates torques that speed up the loop and slow down the disk. But these torques are internal to the combined disk + loop system, so $\tau_{net} = 0$ and the *total* angular momentum of the disk + loop system is conserved.

VISUALIZE **FIGURE 12.56** is a before-and-after pictorial representation. Initially only the disk is rotating, at angular velocity $\vec{\omega}_i$. The rotation is about an axis of symmetry, so the angular momentum $\vec{L} = I\vec{\omega}$ is parallel to $\vec{\omega}$. At the end of the problem, $\vec{\omega}_{disk} = \vec{\omega}_{loop} = \vec{\omega}_f$.

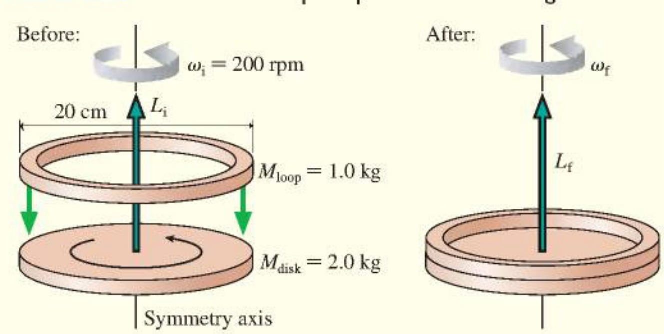
SOLVE Both angular momentum vectors point along the rotation axis. Conservation of angular momentum tells us that the magnitude of $\vec{L}$ is unchanged. Thus

$$L_f = I_{disk}\omega_f + I_{loop}\omega_f = L_i = I_{disk}\omega_i$$

Solving for ω_f gives

$$\omega_f = \frac{I_{disk}}{I_{disk} + I_{loop}}\omega_i$$

FIGURE 12.56 The circular loop drops onto the rotating disk.



The moments of inertia for a disk and a loop can be found in Table 12.2, leading to

$$\omega_f = \frac{\frac{1}{2}M_{disk}R^2}{\frac{1}{2}M_{disk}R^2 + M_{loop}R^2}\omega_i = 100 \text{ rpm}$$

ASSESS The angular velocity has been reduced to half its initial value, which seems reasonable.

STOP TO THINK 12.7 Two buckets spin around in a horizontal circle on frictionless bearings. Suddenly, it starts to rain. As a result,

a. The buckets continue to rotate at constant angular velocity because the rain is falling vertically while the buckets move in a horizontal plane.
b. The buckets continue to rotate at constant angular velocity because the total mechanical energy of the bucket + rain system is conserved.
c. The buckets speed up because the potential energy of the rain is transformed into kinetic energy.
d. The buckets slow down because the angular momentum of the bucket + rain system is conserved.
e. Both a and b.
f. None of the above.

12.12 ADVANCED TOPIC Precession of a Gyroscope

Rotating objects can exhibit surprising and unexpected behaviors. For example, a common lecture demonstration makes use of a bicycle wheel with two handles along the axis. The wheel is spun, then handed to an unsuspecting student who is asked to turn the spinning wheel 90°. Surprisingly, this is *very hard to do*. The reason is that the angular momentum is a *vector*, so the wheel's rotation axis—the direction of $\vec{L}$—is highly resistant to change. If the wheel is spinning fast, a *large* torque is required to turn the wheel's axis.

We'll look at a related example: the precession of a gyroscope. A **gyroscope**—whether it's a toy or a precision instrument used for navigation—is a rapidly spinning wheel or disk whose axis of rotation can assume any orientation. As it spins, it has angular momentum $\vec{L} = I\vec{\omega}$ along the rotation axis. A navigation gyroscope is mounted in gimbals that allow it to spin with virtually no torque from the environment. Once its axis is pointed north, conservation of angular momentum will ensure that the axis continues to point north no matter how the ship or plane moves.

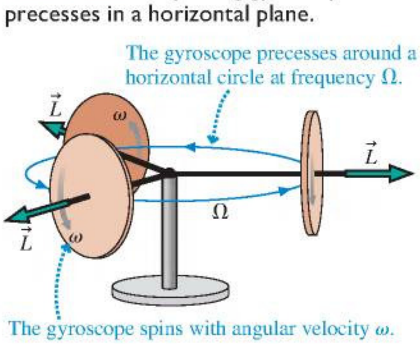


FIGURE 12.57 A spinning gyroscope precesses in a horizontal plane.

We want to consider a horizontal gyroscope, with the disk spinning in a vertical plane, that is supported at only one end of its axle, as shown in FIGURE 12.57. You would expect it to simply fall over—but it doesn't. Instead, the axle remains horizontal, parallel to the ground, while the entire gyroscope slowly rotates in a horizontal plane. This steady change in the orientation of the rotation axis is called **precession,** and we say that the gyroscope precesses about its point of support. The **precession frequency** Ω (capital Greek omega) is much less than the disk's rotation frequency ω. Note that Ω, like ω, is in rad/s.

You might object that angular momentum is not conserved during precession. This is true. The *magnitude* of $\vec{L}$ is constant, but its *direction* is changing. However, angular momentum is conserved only for an isolated system, one on which there is no net torque. The spinning gyroscope is *not* an isolated system because gravity is exerting a torque on it. Indeed, understanding the relationship between the gravitational torque and the angular momentum is the key to understanding why the gyroscope precesses.

FIGURE 12.58a shows a gyroscope that is *not* spinning. When released, it most definitely falls over by rotating about the point of support until the disk hits the table. Because the motion is *rotation*, rather than the translational motion of a gyroscope that is simply dropped, we can analyze it using the concepts of torque and angular momentum.

There are two forces acting on the gyroscope: gravity pulling downward at the disk's center of mass (we'll assume that the axle is massless) and the normal force of the support pushing upward. The normal force exerts no torque about the pivot point because it acts at the pivot point, so **the net torque on the gyroscope is entirely a gravitational torque:**

$$\vec{\tau} = \vec{r} \times \vec{F}_G = Mgd\,\hat{\imath} \tag{12.56}$$

where $d = |\vec{r}\,|$ is the distance from the pivot to the center of the disk. To evaluate the cross product, we redrew $\vec{F}_G$ at the pivot and established a coordinate system with the z-axis along the axle. Vectors $\vec{r}$ and $\vec{F}_G$ are perpendicular ($\sin\alpha = 1$), and by using the right-hand rule we see that $\vec{\tau}$ points along the x-axis.

We found in the previous section that a torque causes the angular momentum to change. In particular,

$$\frac{d\vec{L}}{dt} = \vec{\tau} \tag{12.57}$$

So in a small interval of time dt, the torque causes the gyroscope's angular momentum about the point of support to change by $d\vec{L} = \vec{\tau}\,dt$.

FIGURE 12.58b shows graphically what happens. Initially, when the gyroscope is first released, $\vec{L} = \vec{0}$. After a small interval of time, the gyroscope acquires a small amount of angular momentum $d\vec{L}$ in the direction of $\vec{\tau}$, the $\hat{\imath}$ direction. An angular momentum along the x-axis means that the gyroscope is rotating in the yz-plane—which is exactly what it does as it starts to fall. During the next interval of time, $\vec{L}$ increases a bit more in the $\hat{\imath}$ direction, and then a bit more. This is what we expect as the falling gyroscope picks up speed, increasing its angular momentum.

Now the magnitude of $\vec{\tau}$ does not remain constant—the angle between $\vec{r}$ and $\vec{F}_G$ changes as the gyroscope falls, changing the cross product—so integrating Equation 12.57 symbolically is very difficult. Nonetheless, the *direction* of $\vec{\tau}$ is always the $\hat{\imath}$ direction, so we can see that the angular momentum keeps increasing in the $\hat{\imath}$ direction as the gyroscope falls.

What's different about a spinning gyroscope that causes it to precess rather than fall? In FIGURE 12.59a we've again just released the gyroscope, its axle is again along the z-axis, but now it's spinning with angular velocity $\vec{\omega} = \omega\hat{k}$. Consequently, the gyroscope has initial angular momentum $\vec{L} = I\vec{\omega} = I\omega\hat{k}$ along the z-axis. The torque is exactly as we calculated above, and that torque again causes the angular momentum to change by $d\vec{L} = \vec{\tau}\,dt$. **The only difference is that the gyroscope starts with initial angular momentum**—but that makes all the difference.

FIGURE 12.59b, looking down from above, shows the initial angular momentum $\vec{L}$. A very small time interval dt after we release the gyroscope, its angular momentum will have changed to $\vec{L} + d\vec{L}$. The small *change* in angular momentum, $d\vec{L}$, is parallel to the torque and thus *perpendicular* to the spinning gyroscope's angular momentum $\vec{L}$. Because we're adding vectors, not scalars, the "new" angular momentum has rotated to a new position but *not* increased in magnitude. So during dt, the angular momentum—and thus the entire gyroscope—rotates through a small angle $d\phi$ in the horizontal plane.

The torque vector is always perpendicular to the axle, and thus $d\vec{L}$ is always perpendicular to $\vec{L}$. With each subsequent time interval dt, the gyroscope rotates through another small angle $d\phi$ while the magnitude of the angular momentum (and hence the disk's angular velocity ω) is unchanged. The gyroscope is precessing in the horizontal plane!

You've encountered a similar situation previously. If a ball is initially at rest, pulling on it with a string causes the ball to accelerate (increasing $\vec{v}$) in the direction of the pull. That is, $\vec{v}$ increases in magnitude but doesn't change direction. But if a ball on a string is in uniform circular motion, a force directed to the center—the string tension—has a very different effect. $d\vec{v}$ points toward the center, because that's the direction of the centripetal acceleration, but now $d\vec{v}$ is perpendicular to $\vec{v}$. Adding them as vectors to get $\vec{v} + d\vec{v}$ changes the *direction* of the velocity vector but not its magnitude. Then, as now, having an initial vector ($\vec{v}$ or $\vec{L}$) leads to very different behavior than not having an initial vector.

The small horizontal rotation $d\phi$ is a small piece of the precession. Because it occurs during the small time interval dt, the *rate* of horizontal rotation—the precession frequency—is

$$\Omega = \frac{d\phi}{dt} \tag{12.58}$$

FIGURE 12.58 The gravitational torque on a nonspinning gyroscope causes it to fall over.

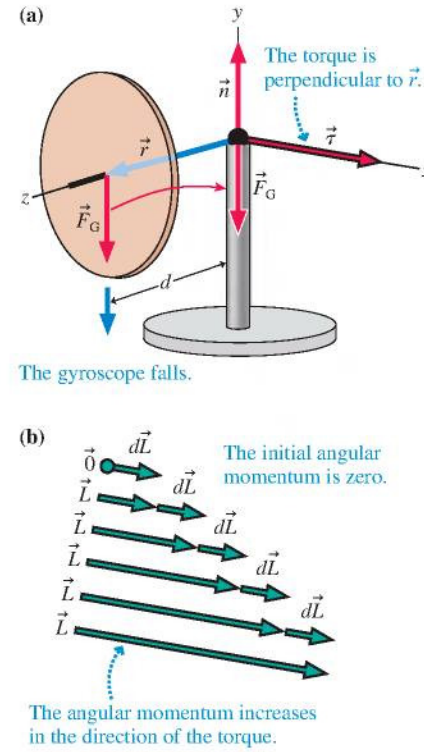
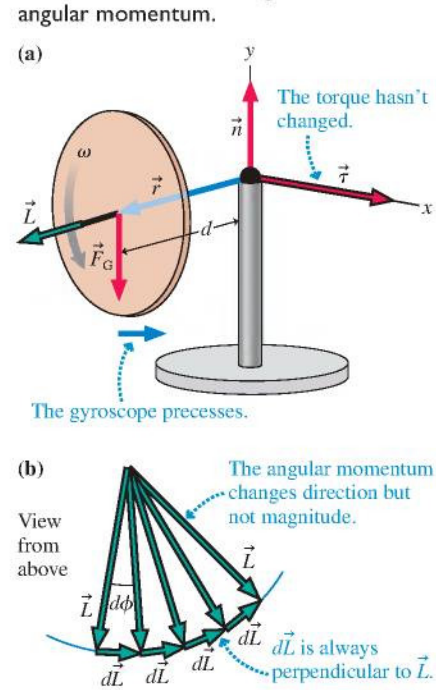


FIGURE 12.59 For a spinning gyroscope, the gravitational torque changes the direction but not the magnitude of the angular momentum.



In Figure 12.59b the small length dL is the arc length spanned by $d\phi$, hence $d\phi = dL/L$ and the precession frequency is

$$\Omega = \frac{dL/L}{dt} = \frac{dL/dt}{L} \tag{12.59}$$

From Equations 12.56 and 12.57, $dL/dt = \tau = Mgd$. Further, the gyroscope's angular momentum has magnitude $L = I\omega$, where I is the moment of inertia of the disk rotating about the axle. Thus the precession frequency of the gyroscope—in rad/s—is

$$\Omega = \frac{Mgd}{I\omega} \tag{12.60}$$

Because the spin angular velocity ω is in the denominator, a very rapidly spinning gyroscope precesses very slowly. As the gyroscope runs down, due to any little bit of friction, it begins to process faster and faster.

We have made one tacit assumption. As the gyroscope precesses, the precessional motion has its own angular momentum along the vertical axis. The gyroscope's angular momentum $\vec{L}$ is not simply the angular momentum of the spinning disk, as we assumed, but the vector sum $\vec{L}_{spin} + \vec{L}_{precess}$. As long as the gyroscope precesses slowly, with $\Omega \ll \omega$, the precessional angular momentum is very small compared to the spin angular momentum and our assumption is well justified. But toward the end of the gyroscope's motion, as ω decreases and Ω increases, our model of precession breaks down and the gyroscope's motion becomes more complex.

EXAMPLE 12.20 A precessing gyroscope

A gyroscope used in a lecture demonstration consists of a 120 g, 7.0-cm-diameter solid disk that rotates on a lightweight axle. From the center of the disk to the end of the axle is 5.0 cm. When spun, placed on a stand, and released, the gyroscope is observed to precess with a period of 1.0 s. How fast, in rpm, is it spinning?

SOLVE The precession frequency is given by Equation 12.60. The moment of inertia of a disk of mass M and radius R about an axis through its center is $I = \frac{1}{2}MR^2$. Inserting this into Equation 12.60, we see that the precession frequency

$$\Omega = \frac{Mgd}{I\omega} = \frac{Mgd}{\frac{1}{2}MR^2\omega} = \frac{2gd}{\omega R^2}$$

is actually independent of the gyroscope's mass. Solving for ω gives

$$\omega = \frac{2gd}{\Omega R^2}$$

A precession period of 1.0 s corresponds to the precession frequency

$$\Omega = \frac{2\pi \text{ rad}}{1.0 \text{ s}} = 6.28 \text{ rad/s}$$

Thus the gyroscope's spin angular velocity is

$$\omega = \frac{2(9.80 \text{ m/s}^2)(0.050 \text{ m})}{(6.28 \text{ rad/s})(0.035 \text{ m})^2} = 127 \text{ rad/s}$$

Converting to rpm gives

$$\omega = 127 \text{ rad/s} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} \times \frac{60 \text{ s}}{1 \text{ min}} = 1200 \text{ rpm}$$

ASSESS 1200 rpm is 20 rev/s. That seems reasonable for a spinning top or gyroscope. And $\Omega \ll \omega$, so our precession model of the gyroscope is valid.

CHALLENGE EXAMPLE 12.21 The ballistic pendulum revisited

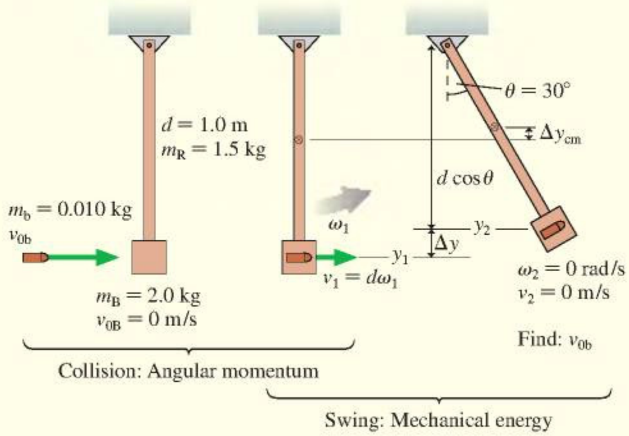
A 2.0 kg block hangs from the end of a 1.5 kg,1.0-m-long rod, together forming a pendulum that swings from a frictionless pivot at the top end of the rod. A 10 g bullet is fired horizontally into the block, where it sticks, causing the pendulum to swing out to a 30° angle. What was the speed of the bullet?

MODEL Model the rod as a uniform rod that can rotate around one end, and assume the block is small enough to model as a particle. There are no external torques on the bullet + block + rod system, so angular momentum is conserved in the inelastic collision. Further, the mechanical energy of the system is conserved after (but not during) the collision as the pendulum swings outward.

VISUALIZE **FIGURE 12.60** is a pictorial representation. This is a two-part problem, so we've separated the collision's before-and-after from the pendulum swing's before-and-after. The end of the collision is the beginning of the swing.

SOLVE This is a *ballistic pendulum*. Example 11.5 considered a simpler ballistic pendulum with a mass on a string, rather than on

FIGURE 12.60 Pictorial representation of the bullet hitting the pendulum.



a rod, and a review of that example is highly recommended. The key to both is that a different conservation law applies to each part of the problem.

Angular momentum is conserved in the collision, thus $L_1 = L_0$. Before the collision, the angular momentum—which we'll measure about the pendulum's pivot point—is entirely that of the bullet. The angular momentum of a particle is $L = mrv \sin \beta$. An instant before the collision, just as the bullet reaches the block, $r = d$ and, because $\vec{v}$ is perpendicular to $\vec{r}$ at that instant, $\beta = 90°$. Thus $L_0 = m_b d v_{0b}$. (This is the magnitude of the angular momentum; from the right-hand rule, the angular momentum vector points out of the page.)

An instant after the collision, but before the pendulum has had time to move, the rod has angular velocity ω_1 and the block, with the embedded bullet, is moving in a circle with speed $v_1 = \omega_1 r = \omega_1 d$. The angular momentum of the block + bullet system is that of a particle, still with $\beta = 90°$, while that of the rod—an object rotating on a fixed axle—is $I_{rod}\omega_1$. Thus the post-collision angular momentum is

$$L_1 = (m_B + m_b)v_1 r + I_{rod}\omega_1 = (m_B + m_b)\,d^2\omega_1 + \tfrac{1}{3}m_R d^2\omega_1$$

The moment of inertia of the rod was taken from Table 12.2.

Equating the before-and-after angular momenta, then solving for v_{0b}, gives

$$m_b d v_{0b} = (m_B + m_b)d^2\omega_1 + \tfrac{1}{3}m_R d^2\omega_1$$

$$v_{0b} = \frac{m_B + m_b + \frac{1}{3}m_R}{m_b}\,d\omega_1 = 251 d\omega_1$$

Once we know ω_1, which we'll find from energy conservation in the swing, we'll be able to compute the bullet's speed.

Mechanical energy is conserved during the swing, but you must be careful to include all the energies. The kinetic energy has two components: the translational kinetic energy of the block + bullet system and the rotational kinetic energy of the rod. The gravitational potential energy also has two components: the potential energy of the block + bullet system and the potential energy of the rod. The latter changes because the center of mass moves upward as the rod swings. Thus the energy conservation statement is

$$\tfrac{1}{2}(m_B + m_b)v_2^2 + \tfrac{1}{2}I_{rod}\omega_2^2 + (m_B + m_b)gy_2 + m_R g y_{cm2} =$$
$$\tfrac{1}{2}(m_B + m_b)v_1^2 + \tfrac{1}{2}I_{rod}\omega_1^2 + (m_B + m_b)gy_1 + m_R g y_{cm1}$$

Although this looks very complicated, you should convince yourself that we've done nothing more than add up two kinetic energies and two potential energies before and after the swing.

We know that $v_2 = 0$ and $\omega_2 = 0$ at the end of the swing, and that $v_1 = d\omega_1$ at the beginning. We also know the moment of inertia of a rod pivoted at one end. Combining the potential energy terms and using $\Delta y = y_f - y_i$, we thus have

$$\tfrac{1}{2}(m_B + m_b + \tfrac{1}{3}m_R)d^2\omega_1^2 = (m_B + m_b)g\,\Delta y + m_R g\,\Delta y_{cm}$$

We see from Figure 12.60 that the block, at its highest point, is distance $d \cos \theta$ *below* the pivot. It started distance d below the pivot, so the bullet + block system *gained* height $\Delta y = d - d\cos\theta = d(1 - \cos\theta)$. The rod's center of mass started distance $d/2$ below the pivot and rises only half as much as the block, so $\Delta y_{cm} = \frac{1}{2}d(1 - \cos\theta)$. With these, the energy equation becomes

$$\tfrac{1}{2}(m_B + m_b + \tfrac{1}{3}m_R)d^2\omega_1^2 = (m_B + m_b + \tfrac{1}{2}m_R)gd(1 - \cos\theta)$$

We can now solve for ω_1:

$$\omega_1 = \sqrt{\frac{m_B + m_b + \frac{1}{2}m_R}{m_B + m_b + \frac{1}{3}m_R}\,\frac{2g(1 - \cos\theta)}{d}} = 1.70 \text{ rad/s}$$

and with that

$$v_{0b} = 251 d\omega_1 = 430 \text{ m/s}$$

ASSESS 430 m/s seems a reasonable speed for a bullet. This was a challenging problem, but one that you can solve if you focus on the problem-solving strategies—drawing a careful pictorial representation, defining the system, and thinking about which conservation laws apply—rather than hunting for the "right" equation.

SUMMARY

The goal of Chapter 12 has been to understand and apply the physics of rotation.

GENERAL PRINCIPLES

Solving Rotational Dynamics Problems

MODEL Model the object as a **rigid body.**

VISUALIZE Draw a pictorial representation.

SOLVE Use **Newton's second law** for rotational motion:

$$\alpha = \frac{\tau_{net}}{I}$$

Use rotational kinematics to find angles and angular velocities.

ASSESS Is the result reasonable?

Conservation Laws

Energy is conserved for an isolated system.

- Pure rotation $E = K_{rot} + U_G = \frac{1}{2}I\omega^2 + Mgy_{cm}$
- Rolling $E = K_{rot} + K_{cm} + U_G = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_{cm}{}^2 + Mgy_{cm}$

Angular momentum is conserved if $\vec{\tau}_{net} = \vec{0}$.

- Particle $\vec{L} = \vec{r} \times \vec{p}$
- Rotation about a symmetry axis or fixed axle $\vec{L} = I\vec{\omega}$

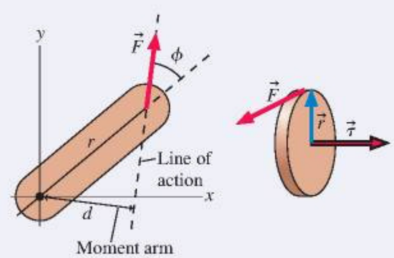
IMPORTANT CONCEPTS

Torque is the rotational equivalent of force:

$$\tau = rF\sin\phi = rF_t = dF$$

The vector description of torque is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

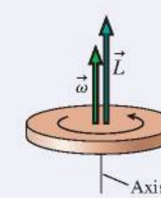


Vector description of rotation

Angular velocity $\vec{\omega}$ points along the rotation axis in the direction of the right-hand rule.

For a rigid body rotating about a fixed axle or an axis of symmetry, the angular momentum is $\vec{L} = I\vec{\omega}$.

Newton's second law is $\dfrac{d\vec{L}}{dt} = \vec{\tau}_{net}$.



A system of particles on which there is no net force undergoes unconstrained rotation about the **center of mass**:

$$x_{cm} = \frac{1}{M}\int x\,dm \qquad y_{cm} = \frac{1}{M}\int y\,dm$$

The gravitational torque on a body can be found by treating the body as a particle with all the mass M concentrated at the center of mass.

The **moment of inertia**

$$I = \sum_i m_i r_i^2 = \int r^2\,dm$$

is the rotational equivalent of mass. The moment of inertia depends on how the mass is distributed around the axis. If I_{cm} is known, I about a parallel axis distance d away is given by the **parallel-axis theorem:** $I = I_{cm} + Md^2$.

APPLICATIONS

Rigid-body model

- Size and shape do not change as the object moves.
- The object is modeled as particle-like atoms connected by massless, rigid rods.

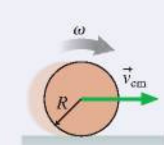
Rigid-body equilibrium

An object is in total equilibrium only if both $\vec{F}_{net} = \vec{0}$ and $\vec{\tau}_{net} = \vec{0}$.

Rolling motion

For an object that rolls without slipping

$$v_{cm} = R\omega$$
$$K = K_{rot} + K_{cm}$$



TERMS AND NOTATION

rigid-body model
rigid body
translational motion
rotational motion
combination motion
center of mass
rotational kinetic energy, K_{rot}
moment of inertia, I
parallel-axis theorem
torque, τ
line of action
moment arm, d
constant-torque model
static equilibrium model
rolling constraint
cross product
vector product
right-hand rule
angular momentum, $\vec{L}$
law of conservation of angular momentum
gyroscope
precession
precession frequency

CONCEPTUAL QUESTIONS

1. Is the center of mass of the dumbbell in **FIGURE Q12.1** at point a, b, or c? Explain.

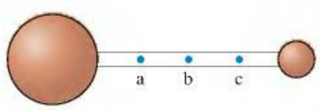


FIGURE Q12.1

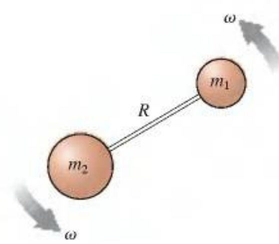


FIGURE Q12.2

2. If the angular velocity ω is held constant, by what *factor* must R change to double the rotational kinetic energy of the dumbbell in **FIGURE Q12.2**?
3. **FIGURE Q12.3** shows three rotating disks, all of equal mass. Rank in order, from largest to smallest, their rotational kinetic energies K_a to K_c.

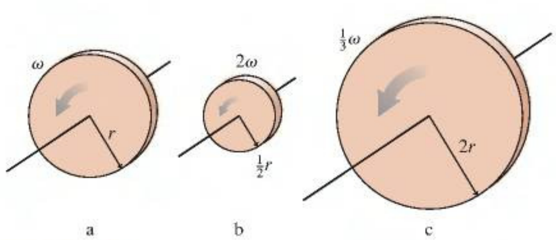


FIGURE Q12.3

4. Must an object be rotating to have a moment of inertia? Explain.
5. The moment of inertia of a uniform rod about an axis through its center is $\frac{1}{12}mL^2$. The moment of inertia about an axis at one end is $\frac{1}{3}mL^2$. Explain *why* the moment of inertia is larger about the end than about the center.
6. You have two solid steel spheres. Sphere 2 has twice the radius of sphere 1. By what *factor* does the moment of inertia I_2 of sphere 2 exceed the moment of inertia I_1 of sphere 1?
7. The professor hands you two spheres. They have the same mass, the same radius, and the same exterior surface. The professor claims that one is a solid sphere and the other is hollow. Can you determine which is which without cutting them open? If so, how? If not, why not?
8. Six forces are applied to the door in **FIGURE Q12.8**. Rank in order, from largest to smallest, the six torques τ_a to τ_f about the hinge on the left. Explain.

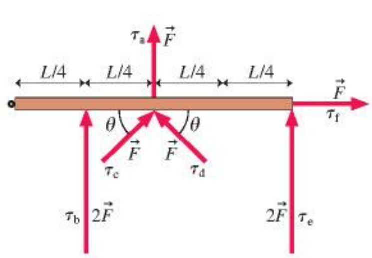


FIGURE Q12.8

9. A student gives a quick push to a ball at the end of a massless, rigid rod, as shown in **FIGURE Q12.9**, causing the ball to rotate clockwise in a *horizontal* circle. The rod's pivot is frictionless.
   a. As the student is pushing, is the torque about the pivot positive, negative, or zero?
   b. After the push has ended, does the ball's angular velocity (i) steadily increase; (ii) increase for awhile, then hold steady; (iii) hold steady; (iv) decrease for awhile, then hold steady; or (v) steadily decrease? Explain.
   c. Right after the push has ended, is the torque positive, negative, or zero?

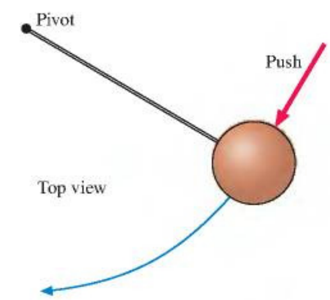


FIGURE Q12.9

10. Rank in order, from largest to smallest, the angular accelerations α_a to α_d in **FIGURE Q12.10**. Explain.

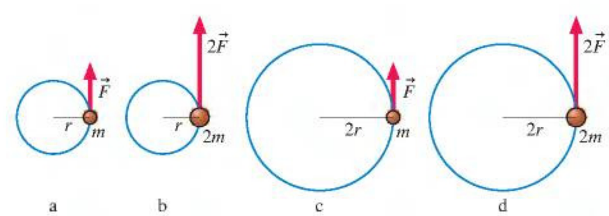


FIGURE Q12.10

11. The solid cylinder and cylindrical shell in **FIGURE Q12.11** have the same mass, same radius, and turn on frictionless, horizontal axles. (The cylindrical shell has lightweight spokes connecting the shell to the axle.) A rope is wrapped around each cylinder and tied to a block. The blocks have the same mass and are held the same height above the ground. Both blocks are released simultaneously. Which hits the ground first? Or is it a tie? Explain.

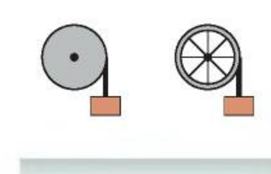


FIGURE Q12.11

12. A diver in the pike position (legs straight, hands on ankles) usually makes only one or one-and-a-half rotations. To make two or three rotations, the diver goes into a tuck position (knees bent, body curled up tight). Why?
13. Is the angular momentum of disk a in **FIGURE Q12.13** larger than, smaller than, or equal to the angular momentum of disk b? Explain.

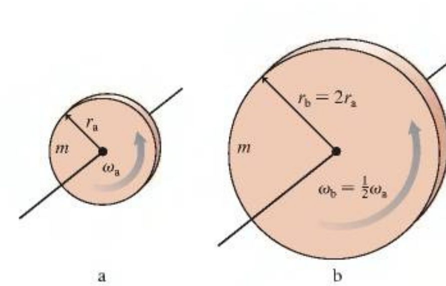


FIGURE Q12.13

EXERCISES AND PROBLEMS

Problems labeled integrate material from earlier chapters.

Exercises

Section 12.1 Rotational Motion

1. || A high-speed drill reaches 2000 rpm in 0.50 s.
   a. What is the drill's angular acceleration?
   b. Through how many revolutions does it turn during this first 0.50 s?
2. || A skater holds her arms outstretched as she spins at 180 rpm. What is the speed of her hands if they are 140 cm apart?
3. || A ceiling fan with 80-cm-diameter blades is turning at 60 rpm. Suppose the fan coasts to a stop 25 s after being turned off.
   a. What is the speed of the tip of a blade 10 s after the fan is turned off?
   b. Through how many revolutions does the fan turn while stopping?
4. ||| An 18-cm-long bicycle crank arm, with a pedal at one end, is attached to a 20-cm-diameter sprocket, the toothed disk around which the chain moves. A cyclist riding this bike increases her pedaling rate from 60 rpm to 90 rpm in 10 s.
   a. What is the tangential acceleration of the pedal?
   b. What length of chain passes over the top of the sprocket during this interval?

Section 12.2 Rotation About the Center of Mass

5. | How far from the center of the earth is the center of mass of the earth + moon system? Data for the earth and moon can be found inside the back cover of the book.
6. | The three masses shown in FIGURE EX12.6 are connected by massless, rigid rods. What are the coordinates of the center of mass?

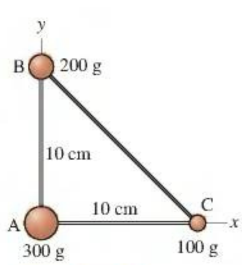


FIGURE EX12.6

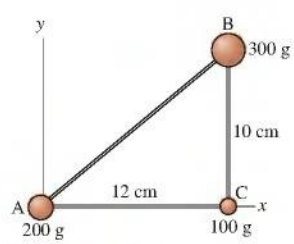


FIGURE EX12.7

7. | The three masses shown in FIGURE EX12.7 are connected by massless, rigid rods. What are the coordinates of the center of mass?
8. || A 100 g ball and a 200 g ball are connected by a 30-cm-long, massless, rigid rod. The balls rotate about their center of mass at 120 rpm. What is the speed of the 100 g ball?

Section 12.3 Rotational Energy

9. || A thin, 100 g disk with a diameter of 8.0 cm rotates about an axis through its center with 0.15 J of kinetic energy. What is the speed of a point on the rim?
10. || What is the rotational kinetic energy of the earth? Assume the earth is a uniform sphere. Data for the earth can be found inside the back cover of the book.
11. || The three 200 g masses in FIGURE EX12.11 are connected by massless, rigid rods.
    a. What is the triangle's moment of inertia about the axis through the center?
    b. What is the triangle's kinetic energy if it rotates about the axis at 5.0 rev/s?

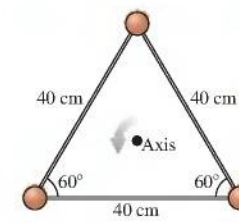


FIGURE EX12.11

12. || A drum major twirls a 96-cm-long, 400 g baton about its center of mass at 100 rpm. What is the baton's rotational kinetic energy?

Section 12.4 Calculating Moment of Inertia

13. || The four masses shown in FIGURE EX12.13 are connected by massless, rigid rods.
    a. Find the coordinates of the center of mass.
    b. Find the moment of inertia about an axis that passes through mass A and is perpendicular to the page.

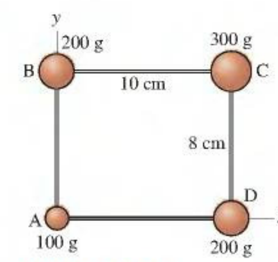


FIGURE EX12.13

14. || The four masses shown in FIGURE EX12.13 are connected by massless, rigid rods.
    a. Find the coordinates of the center of mass.
    b. Find the moment of inertia about a diagonal axis that passes through masses B and D.
15. | The three masses shown in FIGURE EX12.15 are connected by massless, rigid rods.
    a. Find the coordinates of the center of mass.
    b. Find the moment of inertia about an axis that passes through mass A and is perpendicular to the page.
    c. Find the moment of inertia about an axis that passes through masses B and C.

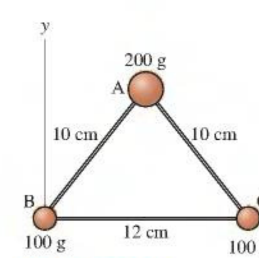


FIGURE EX12.15

16. || A 12-cm-diameter CD has a mass of 21 g. What is the CD's moment of inertia for rotation about a perpendicular axis (a) through its center and (b) through the edge of the disk?
17. || A 25 kg solid door is 220 cm tall, 91 cm wide. What is the door's moment of inertia for (a) rotation on its hinges and (b) rotation about a vertical axis inside the door, 15 cm from one edge?

Section 12.5 Torque

18. || In FIGURE EX12.18, what is the net torque about the axle?

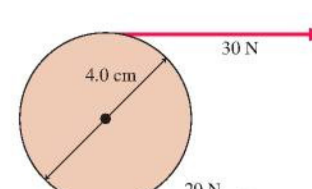


FIGURE EX12.18

19. ‖ In FIGURE EX12.19, what magnitude force provides 5.0 N m net torque about the axle?

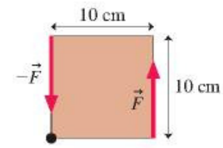


FIGURE EX12.19

20. ‖ The 20-cm-diameter disk in FIGURE EX12.20 can rotate on an axle through its center. What is the net torque about the axle?

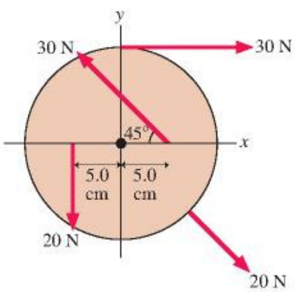


FIGURE EX12.20

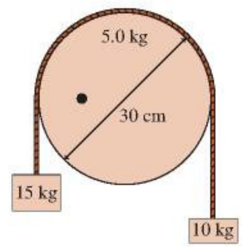


FIGURE EX12.21

21. ‖ The axle in FIGURE EX12.21 is half the distance from the center to the rim. What is the net torque about the axle?

22. ‖ A 4.0-m-long, 500 kg steel beam extends horizontally from the point where it has been bolted to the framework of a new building under construction. A 70 kg construction worker stands at the far end of the beam. What is the magnitude of the torque about the bolt due to the worker and the weight of the beam?

23. ‖ An athlete at the gym holds a 3.0 kg steel ball in his hand. His
BIO arm is 70 cm long and has a mass of 3.8 kg, with the center of mass at 40% of the arm length. What is the magnitude of the torque about his shoulder due to the ball and the weight of his arm if he holds his arm
   a. Straight out to his side, parallel to the floor?
   b. Straight, but 45° below horizontal?

Section 12.6 Rotational Dynamics

Section 12.7 Rotation About a Fixed Axis

24. | An object's moment of inertia is 2.0 kg m^2. Its angular velocity is increasing at the rate of 4.0 rad/s per second. What is the net torque on the object?

25. ‖ An object whose moment of inertia is 4.0 kg m^2 experiences the torque shown in FIGURE EX12.25. What is the object's angular velocity at $t = 3.0$ s? Assume it starts from rest.

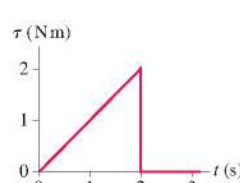


FIGURE EX12.25

26. ‖| A 1.0 kg ball and a 2.0 kg ball are connected by a 1.0-m-long rigid, massless rod. The rod is rotating cw about its center of mass at 20 rpm. What net torque will bring the balls to a halt in 5.0 s?

27. ‖ Starting from rest, a 12-cm-diameter compact disk takes 3.0 s to reach its operating angular velocity of 2000 rpm. Assume that the angular acceleration is constant. The disk's moment of inertia is 2.5×10^{-5} kg m^2.
   a. How much net torque is applied to the disk?
   b. How many revolutions does it make before reaching full speed?

28. ‖ A 4.0 kg, 36-cm-diameter metal disk, initially at rest, can rotate on an axle along its axis. A steady 5.0 N tangential force is applied to the edge of the disk. What is the disk's angular velocity, in rpm, 4.0 s later?

Section 12.8 Static Equilibrium

29. | The two objects in FIGURE EX12.29 are balanced on the pivot. What is distance d?

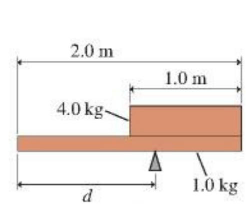


FIGURE EX12.29

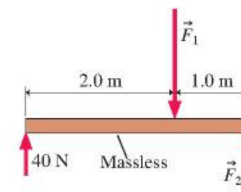


FIGURE EX12.30

30. ‖ The object shown in FIGURE EX12.30 is in equilibrium. What are the magnitudes of $\vec{F}_1$ and $\vec{F}_2$?

31. ‖ The 3.0-m-long, 100 kg rigid beam of FIGURE EX12.31 is supported at each end. An 80 kg student stands 2.0 m from support 1. How much upward force does each support exert on the beam?

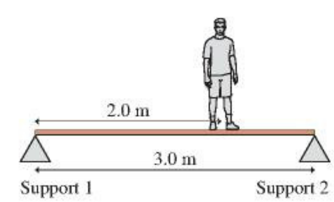


FIGURE EX12.31

32. ‖ A 5.0 kg cat and a 2.0 kg bowl of tuna fish are at opposite ends of the 4.0-m-long seesaw of FIGURE EX12.32. How far to the left of the pivot must a 4.0 kg cat stand to keep the seesaw balanced?

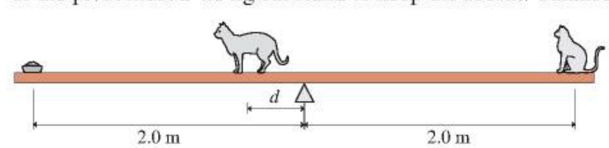


FIGURE EX12.32

Section 12.9 Rolling Motion

33. ‖ A car tire is 60 cm in diameter. The car is traveling at a speed of 20 m/s.
   a. What is the tire's angular velocity, in rpm?
   b. What is the speed of a point at the top edge of the tire?
   c. What is the speed of a point at the bottom edge of the tire?

34. ‖ A 500 g, 8.0-cm-diameter can is filled with uniform, dense food. It rolls across the floor at 1.0 m/s. What is the can's kinetic energy?

35. ‖ An 8.0-cm-diameter, 400 g solid sphere is released from rest at the top of a 2.1-m-long, 25° incline. It rolls, without slipping, to the bottom.
   a. What is the sphere's angular velocity at the bottom of the incline?
   b. What fraction of its kinetic energy is rotational?

36. ‖ A solid sphere of radius R is placed at a height of 30 cm on a 15° slope. It is released and rolls, without slipping, to the bottom. From what height should a circular hoop of radius R be released on the same slope in order to equal the sphere's speed at the bottom?

Section 12.10 The Vector Description of Rotational Motion

37. | Evaluate the cross products $\vec{A} \times \vec{B}$ and $\vec{C} \times \vec{D}$.

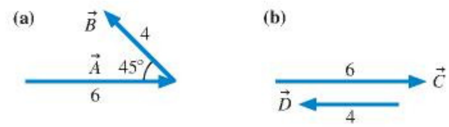


FIGURE EX12.37

38. | Evaluate the cross products $\vec{A} \times \vec{B}$ and $\vec{C} \times \vec{D}$.

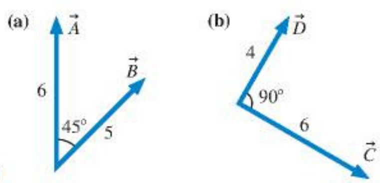


FIGURE EX12.38

39. | Vector $\vec{A} = 3\hat{\imath} + \hat{\jmath}$ and vector $\vec{B} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$. What is the cross product $\vec{A} \times \vec{B}$?

40. || Force $\vec{F} = -10\hat{\jmath}$ N is exerted on a particle at $\vec{r} = (5\hat{\imath} + 5\hat{\jmath})$ m. What is the torque on the particle about the origin?

41. || A 1.3 kg ball on the end of a lightweight rod is located at $(x, y) = (3.0\text{ m}, 2.0\text{ m})$, where the y-axis is vertical. The other end of the rod is attached to a pivot at $(x, y) = (0\text{ m}, 3.0\text{ m})$. What is the torque about the pivot? Write your answer using unit vectors.

Section 12.11 Angular Momentum

42. || What are the magnitude and direction of the angular momentum relative to the origin of the 200 g particle in FIGURE EX12.42?

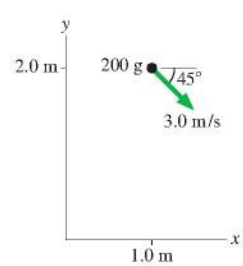


FIGURE EX12.42

43. || What is the angular momentum vector of the 2.0 kg, 4.0-cm-diameter rotating disk in FIGURE EX12.43?

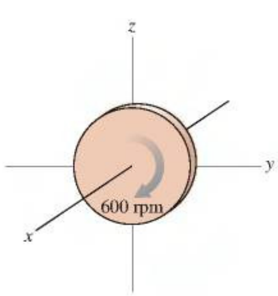


FIGURE EX12.43

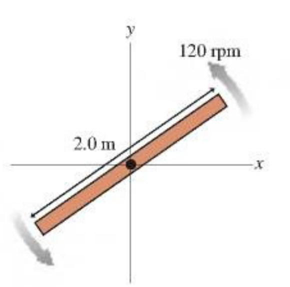


FIGURE EX12.44

44. || What is the angular momentum vector of the 500 g rotating bar in FIGURE EX12.44?

45. || How fast, in rpm, would a 5.0 kg, 22-cm-diameter bowling ball have to spin to have an angular momentum of 0.23 kg m^2/s?

46. || A 2.0 kg, 20-cm-diameter turntable rotates at 100 rpm on frictionless bearings. Two 500 g blocks fall from above, hit the turntable simultaneously at opposite ends of a diameter, and stick. What is the turntable's angular velocity, in rpm, just after this event?

Section 12.12 Precession of a Gyroscope

47. || A 75 g, 6.0-cm-diameter solid spherical top is spun at 1200 rpm on an axle that extends 1.0 cm past the edge of the sphere. The tip of the axle is placed on a support. What is the top's precession frequency in rpm?

48. || A toy gyroscope has a ring of mass M and radius R attached to the axle by lightweight spokes. The end of the axle is distance R from the center of the ring. The gyroscope is spun at angular velocity ω, then the end of the axle is placed on a support that allows the gyroscope to precess.
    a. Find an expression for the precession frequency Ω in terms of M, R, ω, and g.
    b. A 120 g, 8.0-cm-diameter gyroscope is spun at 1000 rpm and allowed to precess. What is the precession period?

Problems

49. || A 300 g ball and a 600 g ball are connected by a 40-cm-long massless, rigid rod. The structure rotates about its center of mass at 100 rpm. What is its rotational kinetic energy?

50. || An 800 g steel plate has the shape of the isosceles triangle
CALC shown in FIGURE P12.50. What are the x- and y-coordinates of the center of mass?
Hint: Divide the triangle into vertical strips of width dx, then relate the mass dm of a strip at position x to the values of x and dx.

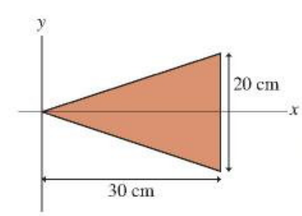


FIGURE P12.50

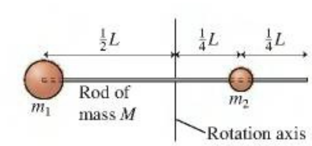


FIGURE P12.51

51. || Determine the moment of inertia about the axis of the object shown in FIGURE P12.51.

52. || What is the moment of inertia of a 2.0 kg, 20-cm-diameter disk for rotation about an axis (a) through the center, and (b) through the edge of the disk?

53. || Calculate by direct integration the moment of inertia for a thin
CALC rod of mass M and length L about an axis located distance d from one end. Confirm that your answer agrees with Table 12.2 when $d = 0$ and when $d = L/2$.

54. || Calculate the moment
CALC of inertia of the rectangular plate in FIGURE P12.54 for rotation about a perpendicular axis through the center.

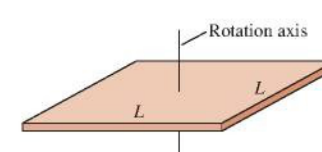


FIGURE P12.54

55. || a. A disk of mass M and radius R has a hole of radius r centered
CALC on the axis. Calculate the moment of inertia of the disk.
    b. Confirm that your answer agrees with Table 12.2 when $r = 0$ and when $r = R$.
    c. A 4.0-cm-diameter disk with a 3.0-cm-diameter hole rolls down a 50-cm-long, 20° ramp. What is its speed at the bottom? What percent is this of the speed of a particle sliding down a frictionless ramp?

56. ||| Consider a solid cone of radius R, height H, and mass M. The volume of a cone is $\frac{1}{3}\pi HR^2$.
CALC
   a. What is the distance from the apex (the point) to the center of mass?
   b. What is the moment of inertia for rotation about the axis of the cone?

   Hint: The moment of inertia can be calculated as the sum of the moments of inertia of lots of small pieces.

57. || A person's center of mass is easily found by having the person lie on a *reaction board*. A horizontal, 2.5-m-long, 6.1 kg reaction board is supported only at the ends, with one end resting on a scale and the other on a pivot. A 60 kg woman lies on the reaction board with her feet over the pivot. The scale reads 25 kg. What is the distance from the woman's feet to her center of mass?
BIO

58. || A 3.0-m-long ladder, as shown in Figure 12.35, leans against a frictionless wall. The coefficient of static friction between the ladder and the floor is 0.40. What is the minimum angle the ladder can make with the floor without slipping?

59. || In FIGURE P12.59, an 80 kg construction worker sits down 2.0 m from the end of a 1450 kg steel beam to eat his lunch. What is the tension in the cable?

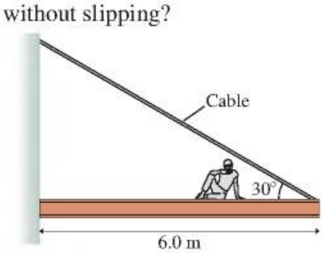


FIGURE P12.59

60. || A 40 kg, 5.0-m-long beam is supported by, but not attached to, the two posts in FIGURE P12.60. A 20 kg boy starts walking along the beam. How close can he get to the right end of the beam without it falling over?

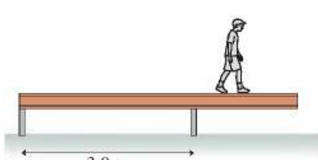


FIGURE P12.60

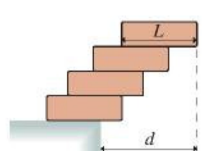


FIGURE P12.61

61. || Your task in a science contest is to stack four identical uniform bricks, each of length L, so that the top brick is as far to the right as possible without the stack falling over. Is it possible, as FIGURE P12.61 shows, to stack the bricks such that no part of the top brick is over the table? Answer this question by determining the maximum possible value of d.

62. || A 120-cm-wide sign hangs from a 5.0 kg, 200-cm-long pole. A cable of negligible mass supports the end of the rod as shown in FIGURE P12.62. What is the maximum mass of the sign if the maximum tension in the cable without breaking is 300 N?

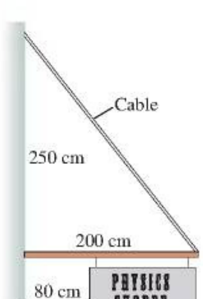


FIGURE P12.62

63. || A piece of modern sculpture consists of an 8.0-m-long, 150 kg stainless steel bar passing diametrically through a 50 kg copper sphere. The center of the sphere is 2.0 m from one end of the bar. To be mounted for display, the bar is oriented vertically, with the copper sphere at the lower end, then tilted 35° from vertical and held in place by one horizontal steel cable attached to the bar 2.0 m from the top end. What is the tension in the cable?

64. ||| Flywheels are large, massive wheels used to store energy. They can be spun up slowly, then the wheel's energy can be released quickly to accomplish a task that demands high power. An industrial flywheel has a 1.5 m diameter and a mass of 250 kg. Its maximum angular velocity is 1200 rpm.
   a. A motor spins up the flywheel with a constant torque of 50 N m. How long does it take the flywheel to reach top speed?
   b. How much energy is stored in the flywheel?
   c. The flywheel is disconnected from the motor and connected to a machine to which it will deliver energy. Half the energy stored in the flywheel is delivered in 2.0 s. What is the average power delivered to the machine?
   d. How much torque does the flywheel exert on the machine?

65. || Blocks of mass m_1 and m_2 are connected by a massless string that passes over the pulley in FIGURE P12.65. The pulley turns on frictionless bearings. Mass m_1 slides on a horizontal, frictionless surface. Mass m_2 is released while the blocks are at rest.
   a. Assume the pulley is massless. Find the acceleration of m_1 and the tension in the string. This is a Chapter 7 review problem.
   b. Suppose the pulley has mass m_p and radius R. Find the acceleration of m_1 and the tensions in the upper and lower portions of the string. Verify that your answers agree with part a if you set $m_p = 0$.

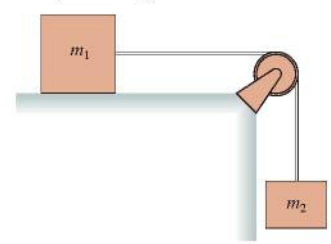


FIGURE P12.65

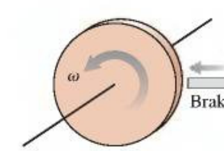


FIGURE P12.66

66. || The 2.0 kg, 30-cm-diameter disk in FIGURE P12.66 is spinning at 300 rpm. How much friction force must the brake apply to the rim to bring the disk to a halt in 3.0 s?

67. || A 30-cm-diameter, 1.2 kg solid turntable rotates on a 1.2-cm-diameter, 450 g shaft at a constant 33 rpm. When you hit the stop switch, a brake pad presses against the shaft and brings the turntable to a halt in 15 seconds. How much friction force does the brake pad apply to the shaft?

68. || Your engineering team has been assigned the task of measuring the properties of a new jet-engine turbine. You've previously determined that the turbine's moment of inertia is 2.6 kg m^2. The next job is to measure the frictional torque of the bearings. Your plan is to run the turbine up to a predetermined rotation speed, cut the power, and time how long it takes the turbine to reduce its rotation speed by 50%. Your data are given in the table. Draw an appropriate graph of the data and, from the slope of the best-fit line, determine the frictional torque.

Rotation (rpm)	Time (s)
1500	19
1800	22
2100	25
2400	30
2700	34

69. ‖ A hollow sphere is rolling along a horizontal floor at 5.0 m/s when it comes to a 30° incline. How far up the incline does it roll before reversing direction?

70. ‖ A 750 g disk and a 760 g ring, both 15 cm in diameter, are rolling along a horizontal surface at 1.5 m/s when they encounter a 15° slope. How far up the slope does each travel before rolling back down?

71. ||| A cylinder of radius R, length L, and mass M is released from rest on a slope inclined at angle θ. It is oriented to roll straight down the slope. If the slope were frictionless, the cylinder would *slide* down the slope without rotating. What minimum coefficient of static friction is needed for the cylinder to roll down without slipping?

72. ‖ The 5.0 kg, 60-cm-diameter disk in FIGURE P12.72 rotates on an axle passing through one edge. The axle is parallel to the floor. The cylinder is held with the center of mass at the same height as the axle, then released.

a. What is the cylinder's initial angular acceleration?

b. What is the cylinder's angular velocity when it is directly below the axle?

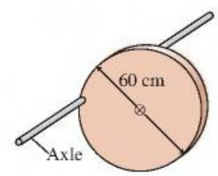


FIGURE P12.72

73. ‖ A thin, uniform rod of length L and mass M is placed vertically on a horizontal table. If tilted ever so slightly, the rod will fall over.

a. What is the speed of the center of mass just as the rod hits the table if there's so much friction that the bottom tip of the rod does not slide?

b. What is the speed of the center of mass just as the rod hits the table if the table is frictionless?

74. ‖ A long, thin rod of mass M and length L is standing straight up on a table. Its lower end rotates on a frictionless pivot. A very slight push causes the rod to fall over. As it hits the table, what are (a) the angular velocity and (b) the speed of the tip of the rod?

75. ||| The marble rolls down the track shown in FIGURE P12.75 and around a loop-the-loop of radius R. The marble has mass m and radius r. What minimum height h must the track have for the marble to make it around the loop-the-loop without falling off?

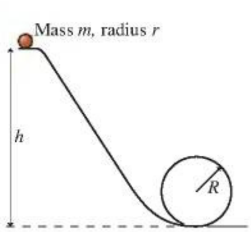


FIGURE P12.75

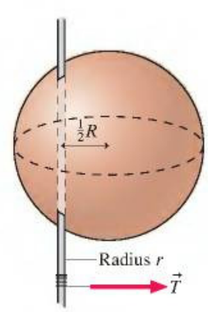


FIGURE P12.76

76. ‖ The sphere of mass M and radius R in FIGURE P12.76 is rigidly attached to a thin rod of radius r that passes through the sphere at distance $\frac{1}{2}R$ from the center. A string wrapped around the rod pulls with tension T. Find an expression for the sphere's angular acceleration. The rod's moment of inertia is negligible.

77. ‖ A satellite follows the elliptical orbit shown in FIGURE P12.77. The only force on the satellite is the gravitational attraction of the planet. The satellite's speed at point a is 8000 m/s.

a. Does the satellite experience any torque about the center of the planet? Explain.

b. What is the satellite's speed at point b?

c. What is the satellite's speed at point c?

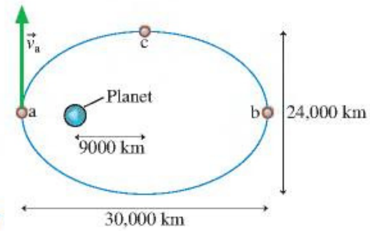


FIGURE P12.77

78. ||| A 10 g bullet traveling at 400 m/s strikes a 10 kg, 1.0-m-wide door at the edge opposite the hinge. The bullet embeds itself in the door, causing the door to swing open. What is the angular velocity of the door just after impact?

79. ‖ A 200 g, 40-cm-diameter turntable rotates on frictionless bearings at 60 rpm. A 20 g block sits at the center of the turntable. A compressed spring shoots the block radially outward along a frictionless groove in the surface of the turntable. What is the turntable's rotation angular velocity when the block reaches the outer edge?

80. ‖ Luc, who is 1.80 m tall and weighs 950 N, is standing at the center of a playground merry-go-round with his arms extended, holding a 4.0 kg dumbbell in each hand. The merry-go-round can be modeled as a 4.0-m-diameter disk with a weight of 1500 N. Luc's body can be modeled as a uniform 40-cm-diameter cylinder with massless arms extending to hands that are 85 cm from his center. The merry-go-round is coasting at a steady 35 rpm when Luc brings his hands in to his chest. Afterward, what is the angular velocity, in rpm, of the merry-go-round?

81. ||| A merry-go-round is a common piece of playground equipment. A 3.0-m-diameter merry-go-round with a mass of 250 kg is spinning at 20 rpm. John runs tangent to the merry-go-round at 5.0 m/s, in the same direction that it is turning, and jumps onto the outer edge. John's mass is 30 kg. What is the merry-go-round's angular velocity, in rpm, after John jumps on?

82. ||| A 45 kg figure skater is spinning on the toes of her skates at 1.0 rev/s. Her arms are outstretched as far as they will go. In this orientation, the skater can be modeled as a cylindrical torso (40 kg, 20 cm average diameter, 160 cm tall) plus two rod-like arms (2.5 kg each, 66 cm long) attached to the outside of the torso. The skater then raises her arms straight above her head, where she appears to be a 45 kg, 20-cm-diameter, 200-cm-tall cylinder. What is her new angular velocity, in rev/s?

83. ||| During most of its lifetime, a star maintains an equilibrium size in which the inward force of gravity on each atom is balanced by an outward pressure force due to the heat of the nuclear reactions in the core. But after all the hydrogen "fuel" is consumed by nuclear fusion, the pressure force drops and the star undergoes a *gravitational collapse* until it becomes a *neutron star*. In a neutron star, the electrons and protons of the atoms are squeezed together by gravity until they fuse into neutrons. Neutron stars spin very rapidly and emit intense pulses of radio and light waves, one pulse per rotation. These "pulsing stars" were discovered in the 1960s and are called *pulsars*.

a. A star with the mass ($M = 2.0 \times 10^{30}$ kg) and size ($R = 7.0 \times 10^8$ m) of our sun rotates once every 30 days. After undergoing gravitational collapse, the star forms a pulsar that is observed by astronomers to emit radio pulses every 0.10 s. By treating the neutron star as a solid sphere, deduce its radius.

b. What is the speed of a point on the equator of the neutron star? Your answers will be somewhat too large because a star cannot be accurately modeled as a solid sphere. Even so, you will be able to show that a star, whose mass is 10^6 larger than the earth's, can be compressed by gravitational forces to a size smaller than a typical state in the United States!

84. || The earth's rotation axis, which is tilted 23.5° from the plane of the earth's orbit, today points to Polaris, the north star. But Polaris has not always been the north star because the earth, like a spinning gyroscope, precesses. That is, a line extending along the earth's rotation axis traces out a 23.5° cone as the earth precesses with a period of 26,000 years. This occurs because the earth is not a perfect sphere. It has an *equatorial bulge*, which allows both the moon and the sun to exert a gravitational torque on the earth. Our expression for the precession frequency of a gyroscope can be written $\Omega = \tau / I\omega$. Although we derived this equation for a specific situation, it's a valid result, differing by at most a constant close to 1, for the precession of any rotating object. What is the average gravitational torque on the earth due to the moon and the sun?

Challenge Problems

85. ||| The bunchberry flower has the fastest-moving parts ever
BIO observed in a plant. Initially, the stamens are held by the petals in a bent position, storing elastic energy like a coiled spring. When the petals release, the tips of the stamen act like medieval catapults, flipping through a 60° angle in just 0.30 ms to launch pollen from anther sacs at their ends. The human eye just sees a burst of pollen; only high-speed photography reveals the details. As FIGURE CP12.85 shows, we can model the stamen tip as a 1.0-mm-long, 10 μg rigid rod with a 10 μg anther sac at the end. Although oversimplifying, we'll assume a constant angular acceleration.

a. How large is the "straightening torque"?

b. What is the speed of the anther sac as it releases its pollen?

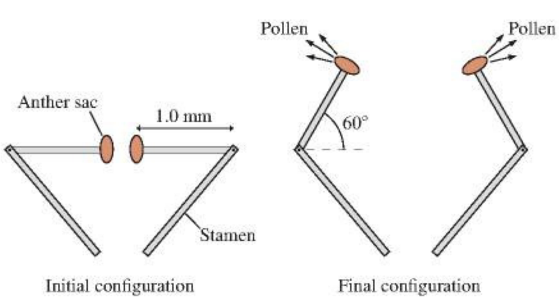


FIGURE CP12.85

86. ||| The two blocks in FIGURE CP12.86 are connected by a massless rope that passes over a pulley. The pulley is 12 cm in diameter and has a mass of 2.0 kg. As the pulley turns, friction at the axle exerts a torque of magnitude 0.50 N m. If the blocks are released from rest, how long does it take the 4.0 kg block to reach the floor?

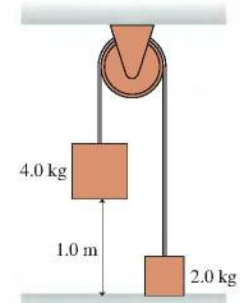


FIGURE CP12.86

87. ||| A rod of length L and mass M has a nonuniform mass distribu-
CALC tion. The linear mass density (mass per length) is $\lambda = cx^2$, where x is measured from the *center* of the rod and c is a constant.

a. What are the units of c?

b. Find an expression for c in terms of L and M.

c. Find an expression in terms of L and M for the moment of inertia of the rod for rotation about an axis through the center.

88. ||| In FIGURE CP12.88, a 200 g toy car is placed on a narrow 60-cm-diameter track with wheel grooves that keep the car going in a circle. The 1.0 kg track is free to turn on a frictionless, vertical axis. The spokes have negligible mass. After the car's switch is turned on, it soon reaches a steady speed of 0.75 m/s relative to the track. What then is the track's angular velocity, in rpm?

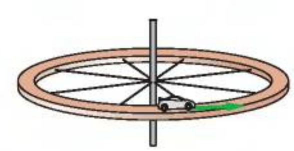


FIGURE CP12.88

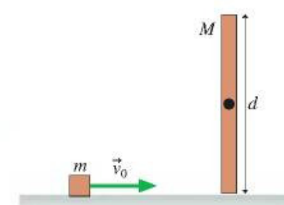


FIGURE CP12.89

89. ||| FIGURE CP12.89 shows a cube of mass m sliding without friction at speed v_0. It undergoes a perfectly elastic collision with the bottom tip of a rod of length d and mass $M = 2m$. The rod is pivoted about a frictionless axle through its center, and initially it hangs straight down and is at rest. What is the cube's velocity—both speed and direction—after the collision?

90. ||| A 75 g, 30-cm-long rod hangs vertically on a frictionless, horizontal axle passing through its center. A 10 g ball of clay traveling horizontally at 2.5 m/s hits and sticks to the very bottom tip of the rod. To what maximum angle, measured from vertical, does the rod (with the attached ball of clay) rotate?